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Linear Mappings Characterized by Action on Zero Products on Banach Algebras

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Abstract. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. In this paper, we characterize certain linear maps $T : \mathcal{A} \longrightarrow X$ by action on zero products. Also we initiate the study of multipliers and Jordan multipliers on standard operator algebras.

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1 Introduction and Preliminaries

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be an $\mathcal{A}$-bimodule. A linear map $T : \mathcal{A} \longrightarrow \mathcal{X}$ is called a *left multiplier* $\big($*right multiplier*$\big)$ if for all $a_1, a_2 \in \mathcal{A}$,

$$T(a_1a_2) = T(a_1)a_2, \qquad \big(T(a_1a_2) = a_1T(a_2)\big),$$

and T is called a left Jordan multiplier $\big($*right Jordan multiplier*$\big)$ if

$$T(a^2) = T(a)a, \big(T(a^2) = aT(a)\big).$$

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The linear map $T:\mathcal{A}\longrightarrow\mathcal{X}$ is called a left (right) derivation if for each $a,b\in\mathcal{A}$,

$$T(ab)=aT(b)+bT(a),\qquad \big(T(ab)=T(a)b+T(b)a\big),$$

and T is called a Jordan right derivation if $T(a\circ b)=2T(a)b+2T(b)a$, where $a\circ b=ab+ba$ is a Jordan product of $a,b\in\mathcal{A}$.

Also T is called a generalized right derivation if there exists an element ξ in $\mathcal{X}$, such that

$$T(ab)=T(a)b+T(b)a-\xi ab,$$

for each $a,b\in\mathcal{A}$, and it is called a generalized Jordan right derivation if there exists an element ξ in $\mathcal{X}$, such that for each $a,b\in\mathcal{A}$,

$$T(a\circ b)=2T(a)b+2T(b)a-\xi(a\circ b).$$

Let X be a Banach space, and $B(X)$ be the operator algebra of all bounded linear operators on X, we denote by $F(X)$, the algebra of all finite rank operators in $B(X)$. Any subalgebra of $B(X)$ which cantains $F(X)$ is called standard operator algebra.

Linear mappings on standard operator algebras is studied in Section 2. We show that under mild conditions, the linear mapping becomes a multiplier or Jordan multiplier.

In Section 3 we consider the subsequent condition on a linear map T from Banach algebra $\mathcal{A}$ into its bimodule $\mathcal{X}$:

$$a,b\in\mathcal{A},\quad ab=ba=0\quad\Longrightarrow\quad aT(b)=0.$$

We investigate whether this condition characterizes multipliers on von Neumann algebras, C^*-algebras, standard operator algebra, or algebras generated by idempotents.

For characterization of linear maps on algebras behaving like left or right multipliers through zero products and different results; see for example [1, 2, 3, 5, 7, 8, 11] and the references therein.

2 Multipliers on Standard Operator Algebras

Let $M_n(\mathbb{C})$ denote the algebra of all $n\times n$ matrices.

Theorem 2.1. *Let T be a linear map from $M_n(\mathbb{C})$ into algebra A such that $T(E) = T(E)E$ holds for all idempotent E in $M_n(\mathbb{C})$. Then T is a Jordan multiplier.*

Proof. Let B be a Hermitian matrix in $M_n(\mathbb{C})$. Then

$$B = \sum_{i=1}^{n} \lambda_i E_i,$$

where $\lambda_i \in \mathbb{C}$ and E_i are idempotents such that for $i \neq j$, $E_iE_j = E_jE_i = 0$. Since for $i \neq j$, $E_i + E_j$ is an idempotent, we get

$$T(E_i + E_j) = T(E_i + E_j)(E_i + E_j).$$

This implies that $T(E_i)E_j + T(E_j)E_i = 0$. Thus, for each Hermitian matrix B we arrive at

$$T(B^2) = T(B)B. \tag{1}$$

Replacing B by $B+C$ where B and C are both Hermitian, we infer that

$$T(BC + CB) = T(B)C + T(C)B. \tag{2}$$

Let H be an arbitrary matrix in $M_n(\mathbb{C})$. Then H can be written in the form $H = B + iC$, where B and C are Hermitian. Hence from (1) and (2), we have $T(H^2) = T(H)H$. Thus, T is a Jordan multiplier. □

In the proof of next theorem we will show that $M_n(\mathbb{C})$ is isomorphic to a subalgebra of $F(X)$ (the algebra of all finite rank operators in $B(X)$). Thus, the product of elements of $M_n(\mathbb{C})$ and $B(X)$ is well defined.

Theorem 2.2. *Let X be a Banach space, $B(X)$ and $F(X)$ be as above. Let $T : F(X) \longrightarrow B(X)$ be a linear map such that $T(E) = T(E)E$ for any idempotent $E \in F(X)$. Then T is a multiplier.*

Proof. Let $A = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus B(X)$. Define a multiplication in A by

$$(t_1, d_1, a_1) \cdot (t_2, d_2, a_2) = (t_1t_2, d_1d_2, a_1d_2).$$

Then A becomes an algebra. Suppose that C is a subalgebra of A generated by all elements of the form $(H, H, T(H))$, for $H \in M_n(\mathbb{C})$.

Now we define a mapping $\alpha : M_n(\mathbb{C}) \longrightarrow C$ by $\alpha(H) = (H, H, T(H))$. By the hypothesis on T we see that α maps idempotents in $M_n(\mathbb{C})$ into idempothents in C. Thus, by [4, Theorem 2.1], α is a Jordan homomorphism, and hence

$$\alpha(HI + IH) = \alpha(H)\alpha(I) + \alpha(I)\alpha(H),$$

where I is the identity element of M_n. By using standard arguments, since $\alpha(I)$ is an idempotent we get $\alpha(H) = \alpha(H)\alpha(I) + \alpha(I)\alpha(H)$. Since the elements of C are in the form $(H, H, T(H))$ for all $H \in M_n$, we conclude that $\alpha(I)$ is a unit element of C. Due to [4, Theorem 2.1], we see that $\alpha = \beta + \gamma$, where $\beta : M_n(\mathbb{C}) \longrightarrow C$ is a homomorphism and $\gamma : M_n(\mathbb{C}) \longrightarrow C$ is an anti-homomorphism. Take $\eta = \beta(I)$ and $\phi = \gamma(I)$. Then η and ϕ are idempotents and $\phi + \eta = \alpha(I)$ is a unit element of C. Therefore $\phi\eta = \eta\phi = 0$, which implies that

$$\gamma(H) = \alpha(H)\phi = \phi\alpha(H) \quad H \in M_n(\mathbb{C}). \tag{3}$$

Since $\phi \in C\phi$, we have $\phi = (P, P, n)$ for some $P \in M_n(\mathbb{C})$, $n \in B(X)$. The relation $\phi^2 = \phi$ yields that

$$P^2 = P, Pn + nP = n \tag{4}$$

By relation (3), ϕ commutes with $\alpha(H)$ for any $H \in M_n$. Hence P commutes with all elements in M_n. This result and (4) tell us that $P = tI$ for some element $t \in \mathbb{C}$. Since γ is an antihomomorphism, (3) implies that $\alpha(WK)\phi = \alpha(K)\alpha(W)\phi$ for $K, W \in M_n$. Therefore from $P = tI$ we conclude that $t(KW - WK) = 0$. Thus $t = 0$, therefore $P = 0$, and so (4) gives us that $n = 0$ too. It follows that $\phi = 0$, and so, $\gamma = 0$. Hence $\alpha = \beta$ is a homomorphism. Due to the definition of θ this implies that T is a multiplier. For $R, S \in F(X)$, there exists an idempotent $P \in F(X)$ such that $PRP = R$ and $PSP = S$. Let $\{y_1, y_2, ..., y_n\}$ be a basis of the range of P. Define linear functionals $g_1, g_2, ..., g_n$ on X by $g_i(y_j) = \delta_{ij}$, $g_i(e) = 0$ for all e in Ker$P$. Let $L \subseteq F(X)$ be the algebra of all operators of the form $l = \sum_{i,j=1}^{n} \lambda_{ij} y_i \otimes g_j$, $\lambda_{ij} \in \mathbb{C}$ and note that L is isomorphic to M_n via the isomorphism $l \longmapsto (\lambda_{ij})$. Thus, for the resteriction of T to L, T becomes a multiplier. Let $x_0 \in X$ and $g_0 \in X^\star$ be chosen such that $g_0(x_0) = 1$. Define operator

$V : X \longrightarrow X$ by $Vx = T(x \otimes g_0)x_0$. For arbitrary $A \in F(X)$, we have $T(Ax \otimes g_0) = T(A)x \otimes g_0$. Applying operator V in this equation to x_0, we get $T(A)x = VAx$, hence $T(A) = VA$. □

Corollary 2.3. *Let $\mathcal{A}$ be a standard operator algebra related to $B(X)$, and $T : \mathcal{A} \longrightarrow B(X)$ be a linear map satisfying $T(E) = T(E)E$ for every idempotent $E \in \mathcal{A}$. If T is continuous with respect to the weak operator topology, then for some $S \in B(X)$ we have $T(A) = SA$, for all $A \in \mathcal{A}$.*

Proof. We know that $F(X)$ is dense in the weak operator topology in every standard operator algebra, since T is continuous, so the result follows from Theorem 2.1. □

3 Characterizing of Derivations and Multipliers

A Banach algebra $\mathcal{A}$ *has the property* $(\mathbb{B})$ if for every continuous bilinear map $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space, the condition

$$a, b \in \mathcal{A}, \quad ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0,$$

implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in \mathcal{A}$.[1]

Proposition 3.1. *Let $\mathcal{A}$ be a unital Banach algebra with property $(\mathbb{B})$, $\mathcal{X}$ be a unital Banach left $\mathcal{A}$-module, and $T : \mathcal{A} \longrightarrow \mathcal{X}$ be a continuous linear operator satisfying*

$$a, b \in \mathcal{A}, \quad ab = ba = 0 \quad \Longrightarrow aT(b) = 0. \tag{5}$$

Then T is a generalized left derivation.

Proof. Let $a_1, b_1 \in \mathcal{A}$ be such that $a_1 b_1 = 0$. Define the bilinear mapping $\psi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ via $\psi(a, b) = b_1 a T(ba_1)$. For every $a, b \in \mathcal{A}$ with $ab = 0$, $b_1 aba_1 = ba_1 b_1 a = 0$, so we have $\psi(a, b) = 0$. Since $\mathcal{A}$ has property $(\mathbb{B})$, for each $a, b, c \in \mathcal{A}$, $\psi(ab, c) = \psi(a, bc)$, i.e.,

$$b_1 abT(ca_1) = b_1 aT(bca_1). \tag{6}$$

For fixed elements $a, b, c \in \mathcal{A}$, we consider the bilinear mapping $\eta : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ such that $\eta(a_1, b_1) = b_1 abT(ca_1) - b_1 aT(bca_1)$, for each $a_1, b_1 \in \mathcal{A}$. Now let $a_1, b_1 \in \mathcal{A}$ with $a_1 b_1 = 0$, then by (6),

$$\eta(a_1, b_1) = 0.$$

Since $\mathcal{A}$ has property $(\mathbb{B})$, it follows that $\eta(a_1b_1, c_1) = \eta(a_1, b_1c_1)$ for all $a_1, b_1, c_1 \in \mathcal{A}$. Thus

$$c_1abT(ca_1b_1) - c_1aT(bca_1b_1) = b_1c_1abT(ca_1) - b_1c_1aT(bca_1).$$

Take $a = c = a_1 = c_1 = e_{\mathcal{A}}$ (the unit element of $\mathcal{A}$), then we obtain

$$T(bb_1) = bT(b_1) + b_1T(b) - b_1bT(e_{\mathcal{A}}),$$

for all $b, b_1 \in \mathcal{A}$. Thus, T is a generalized left derivation. □

It would be interesting to know whether (5) characterizes generalized left derivations or not.

Recall that a C^*-algebra $\mathcal{A}$ is called a W^*-algebra (or von-Neumann algebra) if it is a dual space as a Banach space [6], [10].

Let $\mathcal{I}(A)$ be the set of idempotents of a given Banach algebra $\mathcal{A}$. We say that $\mathcal{A}$ is generated by idempotents if $\mathcal{A} = \overline{\mathfrak{J}(\mathcal{A})}$, where $\mathfrak{J}(\mathcal{A})$ is the subalgebra of $\mathcal{A}$ generated by $\mathcal{I}(A)$.

Theorem 3.2. *Let $\mathcal{A}$ be a W^*-algebra, $\mathcal{X}$ be a unital Banach left $\mathcal{A}$-module, $T : \mathcal{A} \longrightarrow \mathcal{X}$ be a continuous linear operator satisfying condition (5). Then T is a right multiplier.*

Proof. We know that W^*-algebras have property $(\mathbb{B})$. For each idempotent p in $\mathcal{A}$, $p(e_{\mathcal{A}} - p) = (e_{\mathcal{A}} - p)p = 0$, hence $pT(e_{\mathcal{A}} - p) = 0$, so $pT(e_{\mathcal{A}}) = pT(p)$. Let $\mathcal{A}_{sa}$ denote the set of self-adjoint elements of $\mathcal{A}$ and $x \in \mathcal{A}_{sa}$. Then

$$x = \lim_n \sum_{k=1}^{n} \lambda_k p_k,$$

where $\{\lambda_k\}$ are real numbers and $\{p_k\}$ is an orthogonal family of projections in $\mathcal{A}$. Since $p_ip_j = p_jp_i = 0$, for $i \neq j$, condition (5) implies that $p_iT(p_j) = 0$. By Proposition 3.1, T is a generalized left derivation.

Thus for all $x \in \mathcal{A}_{sa}$,

$$\begin{aligned} T(x^2) &= \lim_n T(\sum_{k=1}^n \lambda_k^2 p_k^2) \\ &= \lim_n \sum_{k=1}^n \lambda_k^2 T(p_k^2) = \lim_n \sum_{k=1}^n \lambda_k^2 (2p_k T(p_k) - p_k^2 T(e_{\mathcal{A}})) \\ &= \lim_n \sum_{k=1}^n \lambda_k^2 (2p_k T(e_{\mathcal{A}}) - p_k^2 T(e_{\mathcal{A}})) = \lim_n \sum_{k=1}^n \lambda_k^2 p_k^2 T(e_{\mathcal{A}}) \\ &= x^2 T(e_{\mathcal{A}}). \end{aligned}$$

Thus $T(x^2) = x^2 T(e_{\mathcal{A}})$ for all $x \in \mathcal{A}_{sa}$. It follows from the linearity of T that

$$T(xy + yx) = (xy + yx)T(e_{\mathcal{A}}), \qquad x, y \in \mathcal{A}_{sa}.$$

Now each arbitrary element $a \in \mathcal{A}$ can be written as $a = x + iy$ for $x, y \in \mathcal{A}_{sa}$. Thus

$$\begin{aligned} T(a^2) &= T(x^2 - y^2 + i(xy + yx)) \\ &= x^2 T(e_{\mathcal{A}}) - y^2 T(e_{\mathcal{A}}) + i(xy + yx)T(e_{\mathcal{A}}) \\ &= (x^2 - y^2 + i(xy + yx))T(e_{\mathcal{A}}) = a^2 T(e_{\mathcal{A}}). \end{aligned}$$

Repelacing a by $a + e_{\mathcal{A}}$, we have $T(a + e_{\mathcal{A}})^2 = (a + e_{\mathcal{A}})^2 T(e_{\mathcal{A}})$. Thus $T(a) = aT(e_{\mathcal{A}})$ for all $a \in \mathcal{A}$, and T is a right multiplier. □

Theorem 3.3. *Suppose that $\mathcal{A} = \overline{\mathfrak{J}(\mathcal{A})}$ is a unital Banach algebra and $\mathcal{X}$ is a unital Banach right $\mathcal{A}$-module. Let $T : \mathcal{A} \longrightarrow \mathcal{X}$ be a continuous linear operator satisfying condition (5). Then $T = D + \Phi$, where D is a Jordan right derivation and Φ is a left multiplier.*

Proof. Define bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ by $\phi(a, b) = T(a \circ b) - T(a)b$ for each $a, b \in A$. Then $ab = ba = 0$, implies $\phi(a, b) = 0$. By [3, Lemma 2.2], we have

$$\phi(a, b) + \phi(b, a) = \phi(ab, e_{\mathcal{A}}) + \phi(e_{\mathcal{A}}, ba).$$

Thus,

$$T(a \circ b) - T(a)b + T(b \circ a) - T(b)a = T(ab \circ e_{\mathcal{A}}) - T(ab)e_{\mathcal{A}} + T(e_{\mathcal{A}} \circ ba) - T(e_{\mathcal{A}})ba.$$

So we conclude that

$$T(ab) = T(a)b + T(b)a - T(e_A)ba \tag{7}$$

Settind $a = b$ in (7), we arrive at $T(a^2) = 2T(a)a - T(e_{\mathcal{A}})a^2$.

Define $\Phi : \mathcal{A} \longrightarrow \mathcal{X}$ by $\Phi(a) = T(e_{\mathcal{A}})a$. It is obvious that Φ is a left multiplier. Now define $D : \mathcal{A} \longrightarrow \mathcal{X}$ via

$$D(a) = T(a) - \Phi(a), \qquad a \in \mathcal{A}.$$

So $D(a^2) = T(a^2) - \Phi(a^2)$, thus

$$D(a^2) = 2T(a)a - 2T(e_{\mathcal{A}})a^2. \tag{8}$$

On the other hand,

$$2D(a)a = 2(T(a) - \Phi(a))a = 2T(a)a - 2\Phi(a)a = 2T(a)a - 2T(e_{\mathcal{A}})a^2. \tag{9}$$

From (8), (9), we get $D(a^2) = 2D(a)a$. Thus, D is a Jordan right derivation. This completes the proof. $\square$

Consider the following Lemma from the complex analysis, see ([9]). In order to prove our results we need it.

Lemma 3.4. *Suppose that*

(1) a function f is analytic throughout a domain D;

(2) $f(z) = 0$ at each point z of a domain or line segment contained in D.

Thus $f(z) \equiv 0$ in D; that is $f(z)$ is identically equal to zero throughout D.

For a Banach algebra $\mathcal{A}$ we say that $w \in \mathcal{A}$ is a separating point of $\mathcal{A}$-bimodule $\mathcal{X}$ if the condition $wx = 0$ for all $x \in \mathcal{X}$ implies that $x = 0$.

Theorem 3.5. *Suppose that $\mathcal{A}$ is a unital Banach algebra and $\mathcal{X}$ is a Banach left $\mathcal{A}$-module. Let s be in $Z(\mathcal{A})$ (the center of $\mathcal{A}$), such that s is a separating point of $\mathcal{X}$. Let $T : \mathcal{A} \longrightarrow \mathcal{X}$ be a bounded linear map. Then the following assertions are equivalent.*

(1) $T(ab) = aT(b)$, for all $a, b \in A$ with $ab = ba = s$,

(2) T is a right multiplier.

Proof. First suppose that (1) holds. Then we have

$$T(s) = T(se_{\mathcal{A}}) = sT(e_{\mathcal{A}}). \tag{10}$$

Let $a \in A$ be nonzero. For scalars λ with $|\lambda| < \dfrac{1}{\|a\|}$, $e_{\mathcal{A}} - \lambda a$ is invertible in $\mathcal{A}$. Indeed, $(e_{\mathcal{A}} - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Then

$$\begin{aligned} T(s) &= T[(e_{\mathcal{A}} - \lambda a)^{-1} s (e_{\mathcal{A}} - \lambda a)] = (e_{\mathcal{A}} - \lambda a)^{-1} sT(e_{\mathcal{A}} - \lambda a) \\ &= \sum_{n=0}^{\infty} \lambda^n a^n s(T(e_{\mathcal{A}}) - \lambda T(a) = \sum_{n=0}^{\infty} \lambda^n a^n T(s) - \sum_{n=0}^{\infty} \lambda^{n+1} a^n sT(a) \\ &= \sum_{n=0}^{\infty} \lambda^n a^n T(s) - \sum_{n=1}^{\infty} \lambda^n a^{n-1} sT(a) \\ &= T(s) + \sum_{n=1}^{\infty} \lambda^n a^n T(s) - \sum_{n=1}^{\infty} \lambda^n a^{n-1} T(s). \end{aligned}$$

Thus, for all λ with $|\lambda| < \dfrac{1}{\|a\|}$, we get

$$\sum_{n=1}^{\infty} \lambda^n (a^n T(s) - sa^{n-1} T(a)) = 0. \tag{11}$$

Let $\mu \in \mathcal{A}^*$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be such that

$$f(t) = \mu(\sum_{n=1}^{\infty} t^n (a^n T(s) - sa^{n-1} T(a))),$$

for all $t \in \mathbb{C}$. Due to relation (11) $f(\lambda) = 0$. Now since $f(\lambda) = 0$ for each λ with $|\lambda| < \dfrac{1}{\|a\|}$, by Lemma 3.4, $f(\lambda)$ is identically equal to zero throughout $\mathbb{C}$ and μ arbitrary, Consequently $a^n T(s) - sa^{n-1} T(a) = 0$ for all $n \in \mathbb{N}$. For $n = 1$ we get $aT(s) - sT(a) = 0$, and by using (10), we obtain $asT(e_{\mathcal{A}}) = sT(a)$. Hence $s(aT(e_{\mathcal{A}}) - T(a)) = 0$, since s is a separating point of $\mathcal{X}$ we get $T(a) = aT(e_{\mathcal{A}})$. Thus T is a right multiplier. The converce is clear. $\square$

Proposition 3.6. *Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{X}$ be a Banach left $\mathcal{A}$-module and $\delta : \mathcal{A} \longrightarrow \mathcal{X}$ be a bounded linear operator satisfying $\delta(p^2) = p\delta(p)$ for every projection $p \in \mathcal{A}$. Then δ is a right multiplier.*

Proof. Let $p, q \in \mathcal{A}$ be ortogonal projections. By assumption,

$$\delta((p+q)^2) = (p+q)(\delta(p) + \delta(q)),$$

thus we conclude that $p\delta(q) + q\delta(p) = 0$. Let $a = \sum_{j=1}^{n} \lambda_j p_j$ be a combination of matually orthogonal projections $p_1, p_2, ..., p_n \in \mathcal{A}$. Then

$$p_i\delta(p_j) + p_j\delta(p_i) = 0, \tag{12}$$

for all $i, j \in \{1, 2, ..., n\}$ with $i \neq j$. So

$$\delta(a^2) = \delta(\sum_{j=1}^{n} \lambda_j^2 p_j) = \sum_{j=1}^{n} \lambda_j^2 \delta(p_j). \tag{13}$$

On the other hand, by (12), we have

$$a\delta(a) = \sum_{j=1}^{n} \lambda_j p_j \sum_{i=1}^{n} \lambda_i \delta(p_i) = \lambda_1 p_1 \sum_{i=1}^{n} \lambda_i \delta(p_i) + ... + \lambda_n p_n \sum_{i=1}^{n} \lambda_i \delta(p_i) = \sum_{i=1}^{n} \lambda_i^2 p_i \delta(p_i).$$

It follows from above equation and (13) that $\delta(a^2) = a\delta(a)$. Let $\mathcal{A}_{sa}$ denote the set of self-adjoint elements of $\mathcal{A}$. Simillar to the proof of Theorem 3.2 we deduce that $\delta(a^2) = a\delta(a)$, for all $a \in \mathcal{A}$. Thus δ is a Jordan right multiplier. Due to [11, Theorem 2.3], δ is a right multiplier. □

Proposition 3.7. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach left $\mathcal{A}$-module and $w \in Z(\mathcal{A})$ be a separating point of $\mathcal{X}$. If $\delta : \mathcal{A} \longrightarrow \mathcal{X}$ is a left Jordan multiplier, then $\delta(ba) = \delta(b)a$ for each $a, b \in \mathcal{A}$ with $ab = ba = w$.*

Proof. Suppose that $\delta : \mathcal{A} \longrightarrow \mathcal{X}$ is a left Jordan multiplier. For each $b \in \mathcal{A}$, we have $wb = bw$. Since δ is a left Jordan multiplier, thus $\delta(b^2) = \delta(b)b$. Replacing b by $w + b$, we get $\delta(bw + wb) = \delta(w)b + \delta(b)w$. Therefore

$$2\delta(bw) = \delta(w)b + \delta(b)w. \tag{14}$$

On the other hand, for every Jordan multiplier δ, $\delta(bab) = \delta(b)ab$ for each $a, b \in \mathcal{A}$, thus relation(14) implies that $2\delta(b)ab = \delta(w)b + \delta(b)w$, multiply this equation by a at the right side, we arrive at $2\delta(b)aw = \delta(ab)w + \delta(b)aw$. Hence $\delta(b)aw = \delta(ab)w$. Since w is a separating point of $\mathcal{X}$, we obtain $\delta(ba) = \delta(b)a$ for each $a, b \in \mathcal{A}$ with $ab = ba = w$.

□

In the following theorem we give a complete description of linear mappings δ, τ satisfying $\delta(a)b + \tau(b)a = 0$ for all $a, b \in \mathcal{A}$ with $ab = e_{\mathcal{A}}$.

Theorem 3.8. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{X}$ be a unital Banach right $\mathcal{A}$-bimodule. Let δ, τ be continuous linear mappings from $\mathcal{A}$ into $\mathcal{X}$ satisfying*

$$a, b \in \mathcal{A}, \quad ab = e_{\mathcal{A}} \Longrightarrow \quad \delta(a)b + \tau(b)a = 0.$$

Then δ and τ are generalized Jordan right derivations.

Proof. Let $a = b = e_{\mathcal{A}}$, then $\delta(e_{\mathcal{A}}) + \tau(e_{\mathcal{A}}) = 0$. Let t be an invertible element of $\mathcal{A}$. Then $tt^{-1} = e_{\mathcal{A}}$, so $\delta(t)t^{-1} + \tau(t^{-1})t = 0$, thus $\delta(t) = -\tau(t^{-1})t^2$. Let $a \in \mathcal{A}$, $b = ne_{\mathcal{A}} + a$, then b and $e_{\mathcal{A}} - b$ are invertible in $\mathcal{A}$, for some $n \in \mathbb{N}$, with $n \geq \|a\| + 2$, so we have

$$\begin{aligned}
\delta(b) &= -\tau(b^{-1})b^2 = -\tau(b^{-1}(e_{\mathcal{A}} - b) + e_{\mathcal{A}})b^2 \\
&= -\tau(e_{\mathcal{A}})b^2 - \tau(b^{-1}(e_{\mathcal{A}} - b))b^2 \\
&= -\tau(e_{\mathcal{A}})b^2 - \tau(((e_{\mathcal{A}} - b)^{-1}b)^{-1})b^2 \\
&= -\tau(e_{\mathcal{A}})b^2 - \tau(((e_{\mathcal{A}} - b)^{-1}b)^{-1})((e_{\mathcal{A}} - b)^{-1}b)((e_{\mathcal{A}} - b)^{-1}b))(b^{-1}(e_{\mathcal{A}} - b))(b^{-1}(e_{\mathcal{A}} - b))b^2 \\
&= -\tau(e_{\mathcal{A}})b^2 + \delta((e_{\mathcal{A}} - b)^{-1}b)(b^{-1}(e_{\mathcal{A}} - b))((b^{-1}(e_{\mathcal{A}} - b))b^2 \\
&= -\tau(e_{\mathcal{A}})b^2 + \delta((e_{\mathcal{A}} - b)^{-1}b)(e_{\mathcal{A}} - 2b + b^2) \\
&= -\tau(e_{\mathcal{A}})b^2 + \delta((e_{\mathcal{A}} - b)^{-1} - e_{\mathcal{A}})(e_{\mathcal{A}} - 2b + b^2) \\
&= -\tau(e_{\mathcal{A}})b^2 + \delta((e_{\mathcal{A}} - b)^{-1})(e_{\mathcal{A}} - 2b + b^2) - \delta(e_{\mathcal{A}})(e_{\mathcal{A}} - 2b + b^2) \\
&= -\tau(e_{\mathcal{A}})b^2 - \delta(e_{\mathcal{A}})(e_{\mathcal{A}} - b)^2 - \tau(e_{\mathcal{A}} - b)(e_{\mathcal{A}} - b)^{-1}(e_{\mathcal{A}} - b)^{-1}(e_{\mathcal{A}} - b)^2 \\
&= -\tau(e_{\mathcal{A}})b^2 - \delta(e_{\mathcal{A}})(e_{\mathcal{A}} - b)^2 - \tau(e_{\mathcal{A}} - b) \\
&= -\tau(e_{\mathcal{A}})b^2 - \delta(e_{\mathcal{A}}) + 2\delta(e_{\mathcal{A}})b - \delta(e_{\mathcal{A}})b^2 - \tau(e_{\mathcal{A}}) + \tau(b).
\end{aligned}$$

Therefore $\delta(b) - \tau(b) - 2\delta(e_{\mathcal{A}})b + (\delta(e_{\mathcal{A}}) + \tau(e_{\mathcal{A}}))b^2 = 0$. Thus

$$\delta(b) - \delta(e_{\mathcal{A}})b = \tau(b) - \tau(e_{\mathcal{A}})b.$$

Since $\delta(e_{\mathcal{A}}) + \tau(e_{\mathcal{A}}) = 0$, we have

$$\delta(b) - 2\delta(e_{\mathcal{A}})b = \tau(b), \qquad b \in \mathcal{A}. \tag{15}$$

On the other hand,

$$\delta(b)b^{-1} + \tau(b^{-1})b = 0. \tag{16}$$

Multiple relation (15) by b^{-1} from the right hand side, and applying (16), we arrive at

$$\tau(b)b^{-1} + \tau(b^{-1})b = 2\tau(e_{\mathcal{A}}).$$

For every $b \in \mathcal{A}$, let $|\lambda| \geq \|b\|$ and $\lambda \in \mathbb{C}$. Then $\lambda e_{\mathcal{A}} - b$ is invertible in $\mathcal{A}$ and $(\lambda e_{\mathcal{A}} - b)^{-1} = \sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}}$, so

$$\begin{aligned} 2\tau(e_{\mathcal{A}}) &= \tau(\lambda e_{\mathcal{A}} - b)(\lambda e_{\mathcal{A}} - b)^{-1} + \tau((\lambda e_{\mathcal{A}} - b)^{-1}(\lambda e_{\mathcal{A}} - b)) \\ &= \lambda(\tau(e_{\mathcal{A}}) - \tau(b))(\sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}}) + \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}\tau(b^k)(\lambda e_{\mathcal{A}} - b) \\ &= \lambda\tau(e_{\mathcal{A}})(\sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}}) - \tau(b)(\sum_{k=0}^{\infty} \frac{b^k}{\lambda^{k+1}}) + (\sum_{k=0}^{\infty} \frac{\tau(b^k)}{\lambda^k}) - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}\tau(b^k)b. \end{aligned}$$

Thus

$$2\tau(e_{\mathcal{A}}) = \tau(e_{\mathcal{A}})((\sum_{k=0}^{\infty} \frac{b^k}{\lambda^k}) - \tau(b)\sum_{k=0}^{\infty} \frac{(b^k)}{\lambda^{k+1}} + \sum_{k=0}^{\infty} \frac{\tau(b^k)}{\lambda^k} - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}\tau(b^k)b.$$

Indeed,

$$\begin{aligned} 0 &= \tau(e_{\mathcal{A}})(I + \sum_{k=1}^{\infty} \frac{(b^k)}{\lambda^k} - 2) - \tau(b)\sum_{k=1}^{\infty} \frac{b^{k-1}}{\lambda^k} + \tau(I) + \sum_{k=1}^{\infty} \frac{\tau(b^k}{)}\lambda^k - \sum_{k=1}^{\infty} \frac{1}{\lambda^k}\tau(b^{k-1})b \\ &= \sum_{k=1}^{\infty} \frac{1}{\lambda^k}(\tau(e_{\mathcal{A}})b^k - \tau(b)b^{k-1} + \tau(b^k) - \tau(b^{k-1})b), \end{aligned}$$

for any $|\lambda| \geq \|b\|$. Similar to the proof of Theorem 3.5 we coclude that $\tau(e_{\mathcal{A}})b^k - \tau(b)b^{k-1} + \tau(b^k) - \tau(b^{k-1})b = 0$, for any $k = 2, 3, \ldots$. Let $k = 2$, we have that

$$\tau(e_{\mathcal{A}})b^2 - \tau(b)b + \tau(b^2) - \tau(b)b = 0.$$

Thus,

$$\tau(b^2) = 2\tau(b)b - \tau(e_{\mathcal{A}})b^2. \tag{17}$$

Hence τ is a generalized Jordan right derivation.

Replacing b by b^2 in (15), we get

$$\delta(b^2) = \tau(b^2) - 2\tau(e_{\mathcal{A}})b^2. \tag{18}$$

Hence by (15), (17) and (18) we have $\delta(b^2) = 2\delta(b)b - \delta(e_{\mathcal{A}})b^2$, it means that δ is a generalized Jordan right derivation. $\square$

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