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The Characterization of the Spectrum of a Class of Relations

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Abstract. Hereditary, directed subsets of a group and a semigroup and some of their properties are discussed. A class of relations in terms of the range projections of a partial representation of a discrete group is introduced. It is shown that the spectrum of these relations is homeomorphic to the set of all characters of the diagonal subalgebra of the Toeplitz algebra.

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1. Introduction

In ([3]), the concept of a hereditary, directed subset of a semigroup P is introduced. Also, by a partial representation u of a group G on a Hilbert space H, we mean a map $u : G \longrightarrow B(H)$ with the following properties:

(i) $u_e = 1$

(ii) $u_{t^{-1}} = u_t^*$

(iii) $u_s u_t u_{t^{-1}} = u_{st} u_{t^{-1}}, \;\; s, t \in G.$

Let $u_t u_t^*$ satisfy the special relations $\mathcal{R}$ which will be defined later. The spectrum of the relations $\mathcal{R}$ is defined in ([1]).

On the other hand, Nica, in ([3]), has introduced the spectrum of the diagonal subalgebra of the Toeplitz algebra, denoted by $sp(\mathcal{D})$. In this article, we want to make a homeomorphism between $sp(\mathcal{D})$ and the spectrum of the relations $\mathcal{R}$. For this purpose, first, we bring some terminologies.

A *partially ordered group* is a pair (G, P) where G is a discrete group, and P is a subsemigroup of G. We denote $P^{-1} = \{x^{-1} : x \in P\}$ and always assume that $P \cap P^{-1} = \{e\}$.

For $x, y \in G$, define

$$x \leqslant y \Longleftrightarrow x^{-1} y \in P.$$

The relation $'' \leqslant''$, which is called the *left invariant order relation* induced by P, is a partial order relation. Obviously,

$$P = \{x \in G : e \leqslant x\}, \quad P^{-1} = \{x \in G : x \leqslant e\}.$$

Also, $x \in PP^{-1}$ if and only if x has an upper bound in P.

The ordered group (G, P) is called *quasi-lattice ordered group* if for any $n \geqslant 1$, any $x_1, \cdots, x_n$ in G which have common upper bounds in P, also have a least common upper bounded in P. The least common upper bound of x and y is denoted by $x \vee y$. If $x, y \in G$ have no common upper bound in P, then, by convention, we write $x \vee y = \infty$.

Definition 1.1. *A subset w of G is hereditary if $xP^{-1} \subseteq w$ for every $x \in w$. It is called directed, if every $x, y \in w$ have an upper bound in $w \cap P$.*

We remark that every directed subset of G is contained in PP^{-1}, because every two element in it have an upper bound in P.

Lemma 1.2. *Suppose $w \subseteq G$ is hereditary. Then w is directed if and only if for every $x, y \in w, x \vee y$ exists and is in w.*

Proof. First, suppose that w is directed and take $x, y \in w$. Then there exists an element z in $w \cap P$ such that $x \leqslant z$ and $y \leqslant z$. The quasi-lattice condition implies that the least upper bound of x and $y, x \vee y$, exists and is in P. It remains to prove $x \vee y \in w$. Clearly, $x \vee y \leqslant z$, and so $z^{-1}(x \vee y) \in P^{-1}$, which implies that $x \vee y \in zP^{-1}$. But since w is hereditary and $z \in w$, $zP^{-1} \subseteq w$, and so $x \vee y \in w$.

The converse is clear by taking $z = x \vee y$. □

2. Main Result

Recall that a subset w of P is called *hereditary* if

$$s, t \in P,\ s \leqslant t,\ t \in w \Longrightarrow s \in w.$$

Also, it is called *directed* if any two elements of w have a common upper bound in w. Let Ω denote the set of all nonempty, hereditary directed subsets of P. Consider $w \in \Omega$, and take $t \in w$. Obviously, $e \leqslant t$ and so $e \in w$, because w is hereditary. Furthermore, identifying every subset of P with its characteristic function and considering the product topology on $\{0,1\}^P$, we observe that Ω is a compact, Hausdorff space ([3]).

let (G, P) be a quasi-lattice ordered group. Consider the compact, Hausdorff space $X = \Pi_{t \in G}\{0,1\}$ which can be identified with $\mathcal{P}(G)$, the collection of all subset of G, or with $\{0,1\}^G$. The subset $X_G := \{w \in X : e \in w\}$ is a compact, Hausdorff space with the relative topology inherited from $\{0,1\}^G$. For each $t \in G$, let $X_t = \{w \in X_G : t \in w\}$, and denote the characteristic function on X_t by χ_t.

Define a partial homeomorphism $\theta_t : X_{t^{-1}} \to X_t$ by $\theta_t(w) = tw$. Then $(\{X_t\}_{t \in G}, \{\theta_t\}_{t \in G})$ is a partial action, in the sense of [2] and [4].

Theorem 2.1. *([1]) The set of hereditrary, directed subsets of G containing e, which is denoted by H, is invariant under the partial action θ on X_G; i.e., $\theta_z(H \cap X_{z^{-1}}) \subseteq H$ for every $z \in G$.*

A corollary to this theorem runs as follows:

Corollary 2.2. *Suppose $w \in X_{t^{-1}}$ is hereditary and directed, then so is tw.*

Proof. Clearly, $w \in H$. Since $t^{-1} \in w$, we have $w \in X_{t^{-1}}$. Thus,

$w \in H \cap X_{t^{-1}}$, and so the above theorem implies that

$$tw = \theta_t(w) \in \theta_t(H \cap X_{t^{-1}}) \subseteq H.\square$$

Suppose that the range projections $u_t u_t^{-1} = u_t u_t^*$ of a partial representation u, [1], satisfy the relations $\mathcal{R}$ given by

(i) $u_t^* u_t = 1$, for any $t \in P$;

(ii) $u_t u_t^* u_s u_s^* = u_{t \vee s} u_{t \vee s}^*$, for any $t, s \in G$.

Define the spectrum of the relations $\mathcal{R}$ by

$$\Omega_{\mathcal{R}} = \{w \in X_G : f(t^{-1}w) = 0, \text{ for all } f \in \mathcal{R}, t \in w\}.$$

It is shown that $\Omega_{\mathcal{R}}$ is a compact, Hausdorff space ([1, Proposition 4.1]).

Suppose that $\mathcal{D}$ is the diagonal subalgebra of the Toeplitz algebra $\tau(G,P)$ as introduced in [3]. Indeed, $\mathcal{D}$ consists of all linear operators T on $\ell^2(P)$ whose matrices relative to the canonical basis of $\ell^2(P)$ are diagonal. By the *spectrum of* $\mathcal{D}$, denoted by $sp(\mathcal{D})$, we mean the set of all characters of D. Nica has shown that there is a homeomorphism between $sp(\mathcal{D})$ and Ω. It is worthy of attention to remark that from his homeomorphism, we can obtain the form of each set in Ω; in fact, if $T_t (t \in P)$, are the generators of the Toeplitz algebra then every nonempty, hereditary directed subset of P is of the form

$$A_\gamma = \{t \in P : \gamma(T_t T_t^*) = 1\}$$

where $\gamma \in sp(\mathcal{D})$.

In the remaining, our aim is to identify $\Omega_{\mathcal{R}}$ with $sp(\mathcal{D})$.

Theorem 2.3. *The spaces Ω and $\Omega_{\mathcal{R}}$ are homeomorphic.*

Proof. By Theorem 6.4 of [1], $\Omega_{\mathcal{R}}$ is the set of hereditary, directed subsets of G which contain the identity element. Take $w \in \Omega_{\mathcal{R}}$. Clearly, $w \cap P$ is a nonempty directed subset of P. Suppose $s, t \in P$, $s \leqslant t$, and $t \in w \cap P$. Then $s \in tP^{-1}$, and so $s \in w \cap P$, because w is a hereditary subset of G. Consequently, $w \cap P \in \Omega$ for every $w \in \Omega$. Now, define $\psi : \Omega_{\mathcal{R}} \to \Omega$ by $\psi(w) = w \cap P$. First, we show that ψ is continuous. Suppose that $\{w_i\}_i$ is a net in $\Omega_{\mathcal{R}}$ and $w_i \to w$ is $\Omega_{\mathcal{R}}$ as $i \to \infty$. Identifying each w in X_G with χ_w, the characteristic function of w, we have $x_{w_i} \to x_w$ pointwise as $i \to \infty$, and $\chi_{w_i}\chi_P \to \chi_w\chi_P$ pointwise as $i \to \infty$; that is, $\chi_{w_i \cap P} \to \chi_{w \cap P}$ pointwise as $i \to \infty$; equivalently, $w_i \cap P \to w \cap P$ as $i \to \infty$. Since $\Omega_{\mathcal{R}}$ and Ω are compact Hausdorff spaces, to show that ψ is a homeomorphism, it remains to prove that it is a bijection. So let $w_1, w_2 \in \Omega_{\mathcal{R}}, w_1 \cap P = w_2 \cap P$, but $w_1 \neq w_2$. Assume that $x \in w_1 - w_2$. Since w_1 is a directed subset of G, there exists $z \in w_1 \cap P = w_2 \cap P$ so that $x \leqslant z$, and so $p = x^{-1}z \in P$. Therefore, $zP^{-1} \subseteq w_2$, because w_2 is hereditary. This, in turn, implies that $x = zP^{-1} \in w_2$, which is a contradiction. Hence, ψ is one-to-one. Finally, for every $w' \in \Omega$, consider $w = w'P^{-1}$. Then it can be easily seen that $w \cap P = w'$ and $w \in \Omega_{\mathcal{R}}$. $\square$

Corollary 2.4. *There is a homeomorphism between the spaces* $sp(\mathcal{D})$ *and* $\Omega_{\mathcal{R}}$.

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