The Characterization of the Spectrum of a Class of Relations

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Abstract. Hereditary, directed subsets of a group and a semigroup and some of their properties are discussed. A class of relations in terms of the range projections of a partial representation of a discrete group is introduced. It is shown that the spectrum of these relations is homeomorphic to the set of all characters of the diagonal subalgebra of the Toeplitz algebra.

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1. Introduction

In ([3]), the concept of a hereditary, directed subset of a semigroup $P$ is introduced. Also, by a partial representation $u$ of a group $G$ on a Hilbert space $H$, we mean a map $u : G \rightarrow B(H)$ with the following properties:

(i) $u_e = 1$
(ii) $u_{t^{-1}} = u^*_t$

(iii) $u_s u_t u_{t^{-1}} = u_{st} u_{t^{-1}}, \ s, t \in G$.

Let $u_t u_t^*$ satisfy the special relations $\mathcal{R}$ which will be defined later. The spectrum of the relations $\mathcal{R}$ is defined in ([1]).

On the other hand, Nica, in ([3]), has introduced the spectrum of the diagonal subalgebra of the Toeplitz algebra, denoted by $sp(\mathcal{D})$. In this article, we want to make a homeomorphism between $sp(\mathcal{D})$ and the spectrum of the relations $\mathcal{R}$. For this purpose, first, we bring some terminologies.

A partially ordered group is a pair $(G, P)$ where $G$ is a discrete group, and $P$ is a subsemigroup of $G$. We denote $P^{-1} = \{x^{-1} : x \in P\}$ and always assume that $P \cap P^{-1} = \{e\}$.

For $x, y \in G$, define

$$x \leq y \iff x^{-1}y \in P.$$  

The relation "$\leq"$, which is called the left invariant order relation induced by $P$, is a partial order relation. Obviously,

$$P = \{x \in G : e \leq x\}, \quad P^{-1} = \{x \in G : x \leq e\}.$$  

Also, $x \in PP^{-1}$ if and only if $x$ has an upper bound in $P$.

The ordered group $(G, P)$ is called quasi-lattice ordered group if for any $n \geq 1$, any $x_1, \cdots, x_n$ in $G$ which have common upper bounds in $P$, also have a least common upper bounded in $P$. The least common
upper bound of $x$ and $y$ is denoted by $x \lor y$. If $x, y \in G$ have no common upper bound in $P$, then, by convention, we write $x \lor y = \infty$.

**Definition 1.1.** A subset $w$ of $G$ is hereditary if $xP^{-1} \subseteq w$ for every $x \in w$. It is called directed, if every $x, y \in w$ have an upper bound in $w \cap P$.

We remark that every directed subset of $G$ is contained in $PP^{-1}$, because every two element in it have an upper bound in $P$.

**Lemma 1.2.** Suppose $w \subseteq G$ is hereditary. Then $w$ is directed if and only if for every $x, y \in w, x \lor y$ exists and is in $w$.

**Proof.** First, suppose that $w$ is directed and take $x, y \in w$. Then there exists an element $z$ in $w \cap P$ such that $x \leq z$ and $y \leq z$. The quasi-lattice condition implies that the least upper bound of $x$ and $y, x \lor y$, exists and is in $P$. It remains to prove $x \lor y \in w$. Clearly, $x \lor y \leq z$, and so $z^{-1}(x \lor y) \in P^{-1}$, which implies that $x \lor y \in zP^{-1}$. But since $w$ is hereditary and $z \in w, zP^{-1} \subseteq w$, and so $x \lor y \in w$.

The converse is clear by taking $z = x \lor y$. □

2. Main Result

Recall that a subset $w$ of $P$ is called **hereditary** if

$$s, t \in P, s \leq t, t \in w \implies s \in w.$$
Also, it is called directed if any two elements of \( w \) have a common upper bound in \( w \). Let \( \Omega \) denote the set of all nonempty, hereditary directed subsets of \( P \). Consider \( w \in \Omega \), and take \( t \in w \). Obviously, \( e \leq t \) and so \( e \in w \), because \( w \) is hereditary. Furthermore, identifying every subset of \( P \) with its characteristic function and considering the product topology on \( \{0,1\}^P \), we observe that \( \Omega \) is a compact, Hausdorff space ([3]).

Let \((G,P)\) be a quasi-lattice ordered group. Consider the compact, Hausdorff space \( X = \Pi_{t \in G}\{0,1\} \) which can be identified with \( P(G) \), the collection of all subset of \( G \), or with \( \{0,1\}^G \). The subset \( X_G := \{w \in X : e \in w\} \) is a compact, Hausdorff space with the relative topology inherited from \( \{0,1\}^G \). For each \( t \in G \), let \( X_t = \{w \in X_G : t \in w\} \), and denote the characteristic function on \( X_t \) by \( \chi_t \).

Define a partial homeomorphism \( \theta_t : X_{t^{-1}} \rightarrow X_t \) by \( \theta_t(w) = tw \). Then \((\{X_t\}_{t \in G}, \{\theta_t\}_{t \in G})\) is a partial action, in the sense of [2] and [4].

**Theorem 2.1.** ([1]) The set of hereditary, directed subsets of \( G \) containing \( e \), which is denoted by \( H \), is invariant under the partial action \( \theta \) on \( X_G \); i.e., \( \theta_z(H \cap X_{z^{-1}}) \subseteq H \) for every \( z \in G \).

A corollary to this theorem runs as follows:

**Corollary 2.2.** Suppose \( w \in X_{t^{-1}} \) is hereditary and directed, then so is \( tw \).

**Proof.** Clearly, \( w \in H \). Since \( t^{-1} \in w \), we have \( w \in X_{t^{-1}} \). Thus,
$w \in H \cap X_{t-1}$, and so the above theorem implies that
\[ tw = \theta_t(w) \in \theta_t(H \cap X_{t-1}) \subseteq H. \]$

Suppose that the range projections $u_t u_t^{-1} = u_t u_t^*$ of a partial representation $u$, [1], satisfy the relations $\mathcal{R}$ given by

(i) $u_t^* u_t = 1$, for any $t \in P$;

(ii) $u_t u_t^* u_s u_s^* = u_t \lor_s u_t^* \lor_s$, for any $t, s \in G$.

Define the spectrum of the relations $\mathcal{R}$ by
\[ \Omega_{\mathcal{R}} = \{ w \in X_G : f(t^{-1}w) = 0, \text{ for all } f \in \mathcal{R}, t \in w \}. \]

It is shown that $\Omega_{\mathcal{R}}$ is a compact, Hausdorff space ([1, Proposition 4.1]).

Suppose that $\mathcal{D}$ is the diagonal subalgebra of the Toeplitz algebra $\tau(G, P)$ as introduced in [3]. Indeed, $\mathcal{D}$ consists of all linear operators $T$ on $\ell^2(P)$ whose matrices relative to the canonical basis of $\ell^2(P)$ are diagonal. By the spectrum of $\mathcal{D}$, denoted by $sp(\mathcal{D})$, we mean the set of all characters of $\mathcal{D}$. Nica has shown that there is a homeomorphism between $sp(\mathcal{D})$ and $\Omega$. It is worthy of attention to remark that from his homeomorphism, we can obtain the form of each set in $\Omega$; in fact, if $T_t (t \in P)$, are the generators of the Toeplitz algebra then every nonempty, hereditary directed subset of $P$ is of the form
\[ A_\gamma = \{ t \in P : \gamma(T_t T_t^*) = 1 \} \]

where $\gamma \in sp(\mathcal{D})$. 

In the remaining, our aim is to identify $\Omega_R$ with $sp(D)$.

**Theorem 2.3.** The spaces $\Omega$ and $\Omega_R$ are homeomorphic.

**Proof.** By Theorem 6.4 of [1], $\Omega_R$ is the set of hereditary, directed subsets of $G$ which contain the identity element. Take $w \in \Omega_R$. Clearly, $w \cap P$ is a nonempty directed subset of $P$. Suppose $s, t \in P$, $s \leq t$, and $t \in w \cap P$. Then $s \in tP^{-1}$, and so $s \in w \cap P$, because $w$ is a hereditary subset of $G$. Consequently, $w \cap P \in \Omega$ for every $w \in \Omega$. Now, define $\psi : \Omega_R \to \Omega$ by $\psi(w) = w \cap P$. First, we show that $\psi$ is continuous. Suppose that $\{w_i\}_i$ is a net in $\Omega_R$ and $w_i \to w$ is $\Omega_R$ as $i \to \infty$. Identifying each $w$ in $X_G$ with $\chi_w$, the characteristic function of $w$, we have $x_{w_i} \to x_w$ pointwise as $i \to \infty$, and $\chi_{w_i} \chi P \to \chi_w \chi P$ pointwise as $i \to \infty$; that is, $\chi_{w_i \cap P} \to \chi_{w \cap P}$ pointwise as $i \to \infty$; equivalently, $w_i \cap P \to w \cap P$ as $i \to \infty$. Since $\Omega_R$ and $\Omega$ are compact Hausdorff spaces, to show that $\psi$ is a homeomorphism, it remains to prove that it is a bijection. So let $w_1, w_2 \in \Omega_R$, $w_1 \cap P = w_2 \cap P$, but $w_1 \neq w_2$. Assume that $x \in w_1 - w_2$. Since $w_1$ is a directed subset of $G$, there exists $z \in w_1 \cap P = w_2 \cap P$ so that $x \leq z$, and so $p = x^{-1}z \in P$. Therefore, $zP^{-1} \subseteq w_2$, because $w_2$ is hereditary. This, in turn, implies that $x = zP^{-1} \in w_2$, which is a contradiction. Hence, $\psi$ is one-to-one. Finally, for every $w' \in \Omega$, consider $w = w'P^{-1}$. Then it can be easily seen that $w \cap P = w'$ and $w \in \Omega_R$. □
Corollary 2.4. There is a homeomorphism between the spaces $sp(D)$ and $\Omega_R$.

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