Linear Preservers of Chain Majorization

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Abstract. For \((n \times m)\) matrices \(X, Y \in M_{nm}(\mathbb{R}) (= M_{nm})\), we say \(X\) is chain majorized by \(Y\) and write \(X \ll Y\) if \(X = RY\) where \(R\) is a product of finitely many T-transforms. A linear operator \(T : M_{nm} \rightarrow M_{nm}\) is said to be a linear preserver of the relation \(\ll\) on \(M_{nm}\) if \(X \ll Y\) implies that \(TX \ll TY\). Also, it is said to be strong linear preserver if \(X \ll Y\) is equivalent to \(TX \ll TY\). In this paper we characterize linear and strong linear preservers of \(\ll\).

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1. Introduction

Throughout the paper, the notation \(M_{nm}(\mathbb{R})\) or, simply, \(M_{nm}\) is fixed for the space of all \(n \times m\) real matrices; this is further abbreviated by \(M_n\) when \(m = n\). The space \(M_{n1}\) of all \(n \times 1\) real vectors is denoted by the usual notation \(\mathbb{R}^n\). The collection of all \(n \times n\) permutation matrices is denoted by \(\mathcal{P}(n)\) and the identity matrix is denoted by \(I_n\) or, simply \(I\), if the size \(n\) of the matrix \(I\) is understood from the context.
An $n \times m$ matrix $R = [r_{ij}]$ is called row stochastic if $r_{ij} \geq 0$ and $\sum_{k=1}^{m} r_{ik}$ is equal to 1 for all $i$. A square matrix $D$ is called a doubly stochastic matrix if both $D$ and its transpose $D^t$ are row stochastic matrices. The set of all $n \times n$ doubly stochastic matrices will be denoted by $\mathcal{DS}(n)$.

**Theorem 1.1** (Birkhoff’s Theorem [6]). The totality of all extreme points of the collection of all doubly stochastic matrices is the set of all permutation matrices.

We can describe doubly stochastic matrices by

$$\mathcal{DS}(n) = \{ D \in M_n : D \succeq 0, De = e, D^te = e \},$$

where $e \in \mathbb{R}^n$ is the vector whose components are all $+1$.

Let $X, Y \in M_{nm}$. By a left multivariate majorization $X \prec_{\ellmul} Y$, we mean a relation $X = DY$, for some $n \times n$ doubly stochastic matrix $D$ ([17, p.430]). In this paper, by multivariate majorization we mean left multivariate majorization and show it by $\prec$.

**Definition 1.2** ([20]). A T-transform is a special kind of linear transformation whose matrix has the form $Q = \lambda I + (1-\lambda)S$, with $\lambda \in [0, 1]$ and $S$ a permutation matrix that just interchanges two coordinates.

**Definition 1.3** ([20]). Let $X$ and $Y$ be $n \times m$ matrices. Then $X$ is said to be chain majorized by $Y$, written $X \prec\prec Y$ if $X = RY$ where $R$ is a
product of finitely many $T$-transforms.

**Definition 1.4.** Let $T : M_{nm} \rightarrow M_{nm}$ be a map. $T$ is called a preserver of chain majorization (resp. multivariate majorization) if $X \ll Y$ (resp. $X \prec Y$) is equivalent to $TX \ll TY$ (resp. $TX \prec TY$).

In this paper, by $S$ we mean a transition or a permutation matrix that just interchanges two coordinates, and by $P$ we mean an arbitrary permutation in $P(n)$. In addition, we denote the set of all $T$-transforms by $T_r(n)$. Which is closed under matrix multiplication.

Note that $P(n) \subseteq PT(n) \subseteq DS(n)$ and this fact shows that $X \ll Y$ implies $X \prec Y$ for any $X$ and $Y$ in $M_{nm}$. But the inverse of the above implication is not true (see [17]).

As the reader observes, there is a relation between $\prec$ and $\ll$. So the study of $\prec$ and its preservers can lead us to a characterization of preservers of $\ll$. Some important properties of $\prec$ are stated in the following proposition and many other propositions are stated in [1, 2, 3, 4, 12, 16, 17].

**Proposition 1.5 ([17]).** For each $X, Y, Z \in M_{nm}$ the following assertions hold.

1. $X \prec X$.
2. If $X \prec Y$ and $Y \prec Z$, then $X \prec Z$.
3. If $X \prec Y$, then $X_J \prec Y_J$ for each $k$-tuple $J = (i_1, \ldots, i_k)$ of the set
\{1, \ldots, m\}, where \(X_j\) denotes the matrix whose columns are those of \(X\) with indices \(i_1, \ldots, i_k\).

(4) If \(X \prec Y\) and \(B \in M_{mp}\) for some natural number \(p\), then \(XB \prec YB\).

(5) If \(X \prec Y\) and \(P, Q \in P(n)\), then \(PX \prec QY\).

(6) If \(X \prec Y\), then \(\text{rank}(X) \leq \text{rank}(Y)\).

Remark 1.6. It is easy to prove the above proposition for \(\prec\prec\) instead of \(\prec\).

2. Characterization of Linear Preservers

Before we state the main theorem of this section, we need to recall a theorem in [13], which characterizes the linear preservers of left multivariate majorization.

**Theorem 2.1** ([13]). Let \(T: M_{nm} \to M_{nm}\) be a linear map. Then \(T\) preserves left multivariate majorization if and only if \(T\) has one of the forms i) or ii) as follows:

i) There are \(m\) matrices \(A_1, A_2, \ldots, A_m\) in \(M_{nm}\) such that for any \(X\) in \(M_{nm}\)

\[
TX = \sum_{j=1}^{m} (\sum_{i=1}^{n} X_{ij})A_j;
\]

ii) There are matrices \(L, M\) in \(M_m\) with \(L(L + nM)\) invertible and there
is $P$ in $P(n)$, such that for any $X$ in $M_{nm}$

$$TX = PXL + JXM.$$  

**Theorem 2.2.** Let $T: M_{nm} \to M_{nm}$ be a linear operator. The following assertions are equivalent.

(a) $T$ preserves the chain majorization $≺≺$.

(b) $T$ preserves the multivariate majorization $≺$.

**Proof.** Assume (a) holds, and let $X, Y \in M_{nm}$. Suppose $X \prec Y$. So there exists a doubly stochastic matrix $D \in DS(n)$ such that $X = DY$.

Since $D \in coP(n)$, there are $P_i \in P(n), \alpha_i \in (0, 1]$, with $i = 1, 2, \ldots, k$ such that $\sum_{i=1}^{k} \alpha_i = 1$ and $D = \alpha_i P_i$. In this case, we have

$$TX = T(DY) = T((\sum_{i=1}^{k} \alpha_i P_i)Y) = \sum_{i=1}^{k} T(P_iY)$$

$$= \sum_{i=1}^{k} \alpha_i R_i TY = (\sum_{i=1}^{k} \alpha_i R_i) TY,$$

for some $R_i \in PT(n), i = 1, 2, \ldots, k$ which implies $TX \prec TY$.

Now, let (b) holds and two matrices $X, Y \in M_{nm}$ be such that $X \prec Y$. So there exists a matrix $R \in PT(n)$, where $X = RY$. By Theorem 4.1. there are two possibility forms for the operator $T$. We study them separately and show that in each case we have $TX \prec TY$.

i) Suppose that there are $m$ matrices $A_1, A_2, \ldots, A_m$ in $M_{nm}$ such that for any $X$ in $M_{nm}$

$$TX = \sum_{j=1}^{m} (\sum_{i=1}^{n} X_{ij}) A_j.$$
Then we have

\[ TX = T(RY) = \sum_{j=1}^{m} \sum_{i=1}^{n} ((RY)_{ij} A_j). \]

Since \( R \) is a doubly stochastic matrix, we see

\[ \sum_{i=1}^{n} ((RY)_{ij}) = \sum_{i=1}^{n} R_{ik} Y_{kj} = \sum_{k=1}^{n} Y_{kj} \sum_{i=1}^{n} R_{ik} = \sum_{k=1}^{n} Y_{kj}. \]

Hence

\[ TX = \sum_{j=1}^{m} \sum_{k=1}^{n} Y_{kj} = TY, \]

and this shows that \( TX \prec \prec TY \).

\( i\) Suppose that there are matrices \( L, M \in M_m \) with \( L(L + nM) \) invertible and there is \( P \in P(n) \) such that for any \( X \in M_{nm}, \)

\[ TX = PXL + JXM. \]

So

\[ TX = T(RY) = PRYL + JYRM = P(RYL + RJYM) \]

\[ = PR(YL + JYM) = PRP^t(PYL + JYM) = PRP^tTY. \]

Therefore \( TX \prec \prec TY \), and the proof is complete. \( \square \)

Now we can state the main theorem of this section which is an immediate consequence of Theorem 2.1. and Theorem 2.2.

**Theorem 2.3.** Let \( T : M_{nm} \to M_{nm} \) be a linear map. Then \( T \) preserves chain majorization if and only if \( T \) has one of the forms \( i\) or \( ii\) as follows:
i) There are $m$ matrices $A_1, A_2, \ldots, A_m$ in $M_{nm}$ such that for any $X$ in $M_{nm}$

$$TX = \sum_{j=1}^{m} \left( \sum_{i=1}^{n} X_{ij} \right) A_j;$$

ii) There are matrices $L, M \in M_m$ with $L(L+nM)$ invertible and there is $P \in \mathcal{P}(n)$ such that for any $X \in M_{nm}$

$$TX = PXL + JXM.$$

3. Characterization of Strong Linear Preservers

In this section, we state a characterization of strong preservers of $\prec\prec$ in $M_{nm}$.

**Proposition 3.1.** Let $T: M_{nm} \rightarrow M_{nm}$ be a strong linear preserver of chain majorization. Then $T$ is invertible.

**Proof.** If $TX = 0 = T0$ for some $X \in M_{nm}$, then $TX \prec\prec T0$ and so $X \prec\prec 0$, which is a contradiction. $\square$

**Theorem 3.2 ([10]).** The mapping $T: M_{nm} \rightarrow M_{nm}$ is a strong linear preserver of multivariate majorization if and only if $TX = PXL + JXM$ for some $P \in \mathcal{P}(n)$, and $m \times m$ matrices $L, M$ with $L(L+nM)$ invertible.

**Theorem 3.3.** Let $T: M_{nm} \rightarrow M_{nm}$ be a linear operator. Then the following assertions are equivalent.
(a) \( T \) strongly preserves the chain majorization \( \prec\prec \).

(b) \( T \) strongly preserves the multivariate majorization \( \prec \).

Proof. First assume (a) holds, and \( X, Y \in M_{nm} \). If \( X \prec Y \), then by Theorem 4.2. it follows that \( TX \prec TY \). Now suppose that \( TX \prec TY \) and let \( D \in \mathcal{D}S(n) \) be such that \( TX = DTY \). Since \( D \in \text{co}(\mathcal{P}(n)) \), there are \( P_i \in \mathcal{P}(n), \alpha_i \in (0, 1] \), with \( i = 1, 2, \ldots, k \) such that \( \sum_{i=1}^{k} \alpha_i = 1 \) and \( D = \sum_{i=1}^{k} \alpha_i P_i \). So

\[
TX = DTY = (\sum_{i=1}^{k} \alpha_i P_i)TY = \sum_{i=1}^{k} (\alpha_i P_i TY) = \sum_{i=1}^{k} \alpha_i TX_i = T(\sum_{i=1}^{k} \alpha_i X_i),
\]

where \( TX_i = P_i TY \). The last relation yields \( TX_i \prec \prec TY \), and the hypothesis (a) shows \( X_i \prec \prec Y \), so there are \( R_i \in \mathcal{P}T(n) \) \((i = 1, 2, \ldots, n)\) such that \( X_i = R_i Y \). On the other hand, injectivity of \( T \) and the equation \(TX = T(\sum_{i=1}^{k} \alpha_i X_i)\) show that

\[
X = \sum_{i=1}^{k} \alpha_i X_i = \sum_{i=1}^{k} \alpha_i R_i Y = (\sum_{i=1}^{k} \alpha_i R_i) Y.
\]

And hence \( X \prec Y \).

For a proof of \((b) \Rightarrow (a)\), assume \((b)\) holds and \(X, Y \in M_{nm}\). If \(X \prec \prec Y\) it follows from Theorem 4.2. that \(TX \prec \prec TY\). Now, suppose that \(TX \prec \prec TY\), which implies that \(TX = RTY, TX \prec TY, X \prec Y\) and \(X = DY\) for some \(R \in \mathcal{P}T(n)\) and \(D \in \mathcal{D}S(n)\). On the other hand, the characterization of strong linear preservers of \(\prec\) (Theorem 3.2.) shows
that

\[ PXL + JXM = R(PYL + JYM) = RPYL + JYM \]

for some \( P \in \mathcal{P}(n) \), invertible \( m \times m \) matrix \( L \), and \( m \times m \) matrix \( M \).

The last relation leads us to \( X \prec\prec Y \) as follows:

\[
X + JXNL^{-1} = (P^tRP)Y + JYML^{-1}
\]

\[ \Rightarrow X + JDYNL^{-1} = (P^tRP)Y + JYML^{-1} \]

\[ \Rightarrow X + JYNL^{-1} = (P^tRP)Y + JYML^{-1} \]

\[ \Rightarrow X = (P^tRP)Y. \quad \square \]

Now, we are ready to state the second main result of this paper which is the direct conclusion of Theorem 3.2. and Theorem 3.3.

**Theorem 3.4.** \( T: M_{nm} \rightarrow M_{nm} \) is a strong linear preserver of chain majorization if and only if \( TX = PXL + JXM \) for some \( P \in \mathcal{P}(n) \), and \( m \times m \) matrices \( L, M \) with \( L(L + nM) \) invertible.

**References**


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