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Linear Preservers of Chain Majorization

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Abstract. For ($n \times m$ matrices) $X, Y \in M_{nm}(\mathbb{R})(= M_{nm})$, we say X is chain majorized by Y and write $X \prec\prec Y$ if $X = RY$ where R is a product of finitely many T-transforms. A linear operator $T\colon M_{nm} \to M_{nm}$ is said to be a linear preserver of the relation $\prec\prec$ on M_{nm} if $X \prec\prec Y$ implies that $TX \prec\prec TY$. Also, it is said to be strong linear preserver if $X \prec\prec Y$ is equivalent to $TX \prec\prec TY$. In this paper we characterize linear and strong linear preservers of $\prec\prec$.

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1. Introduction

Throughout the paper, the notation $M_{nm}(\mathbb{R})$ or, simply, M_{nm} is fixed for the space of all $n \times m$ real matrices; this is further abbreviated by M_n when $m = n$. The space M_{n1} of all $n \times 1$ real vectors is denoted by the usual notation $\mathbb{R}^n$. The collection of all $n \times n$ permutation matrices is denoted by $\mathcal{P}(n)$ and the identity matrix is denoted by I_n or, simply I, if the size n of the matrix I is understood from the context.

An $n \times m$ matrix $R = [r_{ij}]$ is called *row stochastic* if $r_{ij} \geqslant 0$ and $\Sigma_{k=1}^{m} r_{ik}$ is equal to 1 for all i. A square matrix D is called a *doubly stochastic* matrix if both D and its transpose D^t are row stochastic matrices. The set of all $n \times n$ doubly stochastic matrices will be denoted by $\mathcal{DS}(n)$.

Theorem 1.1 *(Birkhoff's Theorem [6]). The totality of all extreme points of the collection of all doubly stochastic matrices is the set of all permutation matrices.*

We can describe doubly stochastic matrices by

$$\mathcal{DS}(n) = \{D \in M_n : D \geqslant 0, De = e, D^t e = e\},$$

where $e \in \mathbb{R}^n$ is the vector whose components are all $+1$.
Let $X, Y \in M_{nm}$. By a *left multivariate majorization* $X \prec_{\ell mul} Y$, we mean a relation $X = DY$, for some $n \times n$ doubly stochastic matrix D ([17, p.430]). In this paper, by multivariate majorization we mean left multivariate majorization and show it by $\prec$.

Definition 1.2 ([20]). *A T-transform is a special kind of linear transformation whose matrix has the form $Q = \lambda I + (1 - \lambda)S$, with $\lambda \in [0, 1]$ and S a permutation matrix that just interchanges two coordinates.*

Definition 1.3 ([20]). *Let X and Y be $n \times m$ matrices. Then X is said to be chain majorized by Y, written $X \prec\prec Y$ if $X = RY$ where R is a* *product of finitely many T-transforms.*

Definition 1.4. *Let $T\colon M_{nm} \to M_{nm}$ be a map. T is called a preserver of chain majorization (resp. multivariate majorization) if $X \prec\prec Y$ (resp. $X \prec Y$) is equivalent to $TX \prec\prec TY$ (resp. $TX \prec TY$).*

In this paper, by S we mean a transition or a permutation matrix that just interchanges two coordinates, and by P we mean an arbitrary permutation in $\mathcal{P}(n)$. In addition, we denote the set of all T-transforms by $\mathcal{T}r(n)$. Which is closed under matrix multiplication.
Note that $\mathcal{P}(n) \subseteq \mathcal{PT}(n) \subseteq \mathcal{DS}(n)$ and this fact shows that $X \prec\prec Y$ implies $X \prec Y$ for any X and Y in M_{nm}. But the inverse of the above implication is not true (see [17]).

As the reader observes, there is a relation between $\prec$ and $\prec\prec$. So the study of $\prec$ and its preservers can lead us to a characterization of preservers of $\prec\prec$. Some important properties of $\prec$ are stated in the following proposition and many other propositions are stated in [1, 2, 3, 4, 12, 16, 17].

Proposition 1.5 ([17]). *For each $X, Y, Z \in M_{nm}$ the following assertions hold.*

(1) $X \prec X$.

(2) *If $X \prec Y$ and $Y \prec Z$, then $X \prec Z$.*

(3) *If $X \prec Y$, then $X_J \prec Y_J$ for each k-tuple $J = (i_1, \ldots, i_k)$ of the set*

$\{1,\dots,m\}$, where X_J denotes the matrix whose columns are those of X with indices $i_1,\dots,i_k$.

(4) *If $X \prec Y$ and $B \in M_{mp}$ for some natural number p, then $XB \prec YB$.*

(5) *If $X \prec Y$ and $P, Q \in \mathcal{P}(n)$, then $PX \prec QY$.*

(6) *If $X \prec Y$, then* $\mathrm{rank}(X) \leqslant \mathrm{rank}(Y)$.

Remark 1.6. *It is easy to prove the above proposition for $\prec\prec$ instead of $\prec$.*

2. Characterization of Linear Preservers

Before we state the main theorem of this section, we need to recall a theorem in [13], which characterizes the linear preservers of left multivariate majorization.

Theorem 2.1 ([13]). *Let $T\colon M_{nm} \to M_{nm}$ be a linear map. Then T preserves left multivariate majorization if and only if T has one of the forms i) or ii) as follows:*

i) There are m matrices $A_1, A_2, \dots, A_m$ in M_{nm} such that for any X in M_{nm}

$$TX = \sum_{j=1}^{m}(\sum_{i=1}^{n} Xij)Aj;$$

ii) There are matrices L,M in M_m with $L(L+nM)$ invertible and there *is P in $P(n)$, such that for any X in M_{nm}*

$$TX = PXL + JXM.$$

Theorem 2.2. *Let $T\colon M_{nm} \to M_{nm}$ be a linear operator. The following assertions are equivalent.*

(a) *T preserves the chain majorization $\prec\prec$.*

(b) *T preserves the multivariate majorization $\prec$.*

Proof. Assume (a) holds, and let $X, Y \in M_{nm}$. Suppose $X \prec Y$. So there exists a doubly stochastic matrix $D \in \mathcal{DS}(n)$ such that $X = DY$. Since $D \in co\mathcal{P}(n)$, there are $P_i \in \mathcal{P}(n)$, $\alpha_i \in (0, 1]$, with $i = 1, 2, \dots, k$ such that $\sum_{i=1}^{k} \alpha_i = 1$ and $D = \alpha_i P_i$. In this case, we have

$$TX = T(DY) = T((\sum_{i=1}^{k} \alpha_i P_i)Y) = \sum_{i=1}^{k} T(P_i Y)$$
$$= \textstyle\sum_{i=1}^{k} \alpha_i R_i TY = (\sum_{i=1}^{k} \alpha_i R_i)TY,$$

for some $R_i \in \mathcal{PT}(n)$, $i = 1, 2, \dots, k$ which implies $TX \prec TY$.

Now, let (b) holds and two matrices $X, Y \in M_{nm}$ be such that $X \prec\prec Y$. So there exists a matrix $R \in \mathcal{PT}(n)$, where $X = RY$. By Theorem 4.1. there are two possibility forms for the operator T. We study them separately and show that in each case we have $TX \prec\prec TY$.

i) Suppose that there are m matrices $A_1, A_2, \dots, A_m$ in M_{nm} such that for any X in M_{nm}

$$TX = \sum_{j=1}^{m} (\sum_{i=1}^{n} X_{ij}) A_j.$$

Then we have

$$TX = T(RY) = \sum_{j=1}^{m}\sum_{i=1}^{n}((RY)_{ij}A_j.$$

Since R is a doubly stochastic matrix, we see

$$\sum_{i=1}^{n}((RY)_{ij} = \sum_{i=1}^{n}\sum_{k=1}^{n} R_{ik}Y_{kj} = \sum_{k=1}^{n} Y_{kj}\sum_{i=1}^{n} R_{ik} = \sum_{k=1}^{n} Y_{kj}.$$

Hence

$$TX = \sum_{j=1}^{m}\sum_{k=1}^{n} Y_{kj} = TY,$$

and this shows that $TX \prec\prec TY$.

ii) Suppose that there are matrices $L, M \in M_m$ with $L(L+nM)$ invertible and there is $P \in \mathcal{P}(n)$ such that for any $X \in M_{nm}$,

$$TX = PXL + JXM.$$

So

$$TX = T(RY) = PRYL + JYRM = P(RYL + RJYM)$$

$$= PR(YL + JYM) = PRP^t(PYL + JYM) = PRP^tTY.$$

Therefore $TX \prec\prec TY$, and the proof is complete. □

Now we can state the main theorem of this section which is an immediate consequence of Theorem 2.1. and Theorem 2.2.

Theorem 2.3. *Let $T\colon M_{nm} \to M_{nm}$ be a linear map. Then T preserves chain majorization if and only if T has one of the forms i) or ii) as follows:*

i) There are m matrices $A_1, A_2, \ldots, A_m$ in M_{nm} such that for any X in M_{nm}

$$TX = \sum_{j=1}^{m}(\sum_{i=1}^{n} X_{ij})A_j;$$

ii) There are matrices $L, M \in M_m$ with $L(L+nM)$ invertible and there is $P \in \mathcal{P}(n)$ such that for any $X \in M_{nm}$

$$TX = PXL + JXM.$$

3. Characterization of Strong Linear Preservers

In this section, we state a characterization of strong preservers of $\prec\prec$ in Theorem 3.4.

Proposition 3.1. *Let $T\colon M_{nm} \to M_{nm}$ be a strong linear preserver of chain majorization. Then T is invertible.*

Proof. If $TX = 0 = T0$ for some $X \in M_{nm}$, then $TX \prec\prec T0$ and so $X \prec\prec 0$, which is a contradiction. □

Theorem 3.2 ([10]). *The mapping $T\colon M_{nm} \to M_{nm}$ is a strong linear preserver of multivariate majorization if and only if $TX = PXL + JXM$ for some $P \in \mathcal{P}(n)$, and $m\times m$ matrices L, M with $L(L+nM)$ invertible.*

Theorem 3.3. *Let $T\colon M_{nm} \to M_{nm}$ be a linear operator. Then the following assertions are equivalent.*

(a) T *strongly preserves the chain majorization* $\prec\prec$.

(b) T *strongly preserves the multivariate majorization* $\prec$.

Proof. First assume (a) holds, and $X, Y \in M_{nm}$. If $X \prec Y$, then by Theorem 4.2. it follows that $TX \prec TY$. Now suppose that $TX \prec TY$ and let $D \in \mathcal{DS}(n)$ be such that $TX = DTY$. Since $D \in co(\mathcal{P}(n))$, there are $P_i \in \mathcal{P}(n)$, $\alpha_i \in (0,1]$, with $i = 1, 2, \ldots, k$ such that $\sum_{i=1}^k \alpha_i = 1$ and $D = \sum_{i=1}^k \alpha_i P_i$. So

$$TX = DTY$$
$$= (\sum_{i=1}^k \alpha_i P_i)TY = \sum_{i=1}^k (\alpha_i P_i TY) = \sum_{i=1}^k \alpha_i TX_i = T(\sum_{i=1}^k \alpha_i X_i),$$

where $TX_i = P_i TY$. The last relation yields $TX_i \prec\prec TY$, and the hypothesis (a) shows $X_i \prec\prec Y$, so there are $R_i \in \mathcal{PT}(n)$ $(i = 1, 2, \ldots, n)$ such that $X_i = R_i Y$. On the other hand, injectivity of T and the equation $TX = T(\sum_{i=1}^k \alpha_i X_i)$ show that

$$X = \sum_{i=1}^k \alpha_i X_i = \sum_{i=1}^k \alpha_i R_i Y = (\sum_{i=1}^k \alpha_i R_i)Y.$$

And hence $X \prec Y$.

For a proof of $(b) \Rightarrow (a)$, assume (b) holds and $X, Y \in M_{nm}$. If $X \prec\prec Y$ it follows from Theorem 4.2. that $TX \prec\prec TY$. Now, suppose that $TX \prec\prec TY$, which implies that $TX = RTY$, $TX \prec TY$, $X \prec Y$ and $X = DY$ for some $R \in \mathcal{PT}(n)$ and $D \in \mathcal{DS}(n)$. On the other hand, the characterization of strong linear preservers of $\prec$ (Theorem 3.2.) shows that

$$PXL + JXM = R(PYL + JYM) = RPYL + JYM$$

for some $P \in \mathcal{P}(n)$, invertible $m \times m$ matrix L, and $m \times m$ matrix M. The last relation leads us to $X \prec\prec Y$ as follows:

$$X + JXNL^{-1} = (P^tRP)Y + JYML^{-1}$$

$$\Rightarrow X + JDYNL^{-1} = (P^tRP)Y + JYML^{-1}$$

$$\Rightarrow X + JYNL^{-1} = (P^tRP)Y + JYML^{-1}$$

$$\Rightarrow X = (P^tRP)Y. \square$$

Now, we are ready to state the second main result of this paper which is the direct conclusion of Theorem 3.2. and Theorem 3.3.

Theorem 3.4. *$T\colon M_{nm} \to M_{nm}$ is a strong linear preserver of chain majorization if and only if $TX = PXL + JXM$ for some $P \in \mathcal{P}(n)$, and $m \times m$ matrices L, M with $L(L + nM)$ invertible.*

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