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Convolution Integral Equations with Two Kernels

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Abstract. In this paper, we consider the integral equation of convolution type with some conditions which has two kernels. At first we change this integral equation to a simple operator form and then, we approximate it by Topelitzian and Hankelian series.

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1 Introduction

Integral equation on the half-line with a kernel that depends on the difference of the arguments called Wiener-Hopf Integral Equations. In this paper, we use the notation WHIE for this type of integral equations. This type of integral equations have important and multiple applications in different areas of mathematical natural sciences such as neutron transport theory, nuclear reactor, astrophysics and atmosphere optic [1, 3, 4]. A new factorization method recently found by Yengibaryan [2] leads in many cases to analytical or effective numerical solutions of the WHIE and the integral equation of the convolution type where has the following form

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$$f(x) = g(x) + \int_{-\infty}^{\infty} k(x-t)f(t)dt.$$

Ambartsumian's invariance principle led to a wide and useful application in the theory of WHIE of nonlinear functional equations, Ambartsumian's equations which is in the following form

$$\varphi(s) = 1 + \varphi(s) \int_{a}^{b} \frac{1}{s+p}\varphi(p)d\sigma(p), \qquad (1)$$

plays an important role in effective solution and the mathematical study of special classes of the nonsingular and singular WHIE [1, 4].

This paper is devoted to investigation on the integral equation of the convolution type with two kernels on the entire line in the following form

$$f(x) = g(x) + \int_{0}^{\infty} T_1(x-t)f(t)dt + \int_{-\infty}^{0} T_2(x-t)f(t)dt, \quad x \in \mathbb{R} \qquad (2)$$

where $f \in L_1^{loc}(-\infty, \infty)$ is the unknown function and we know a function is called locally integrable if, around every point in the domain, there is a neighborhood on which the function is integrableand ang $g, T_{1,2}$ are the functions that have the following properties

$$g \in E, \quad 0 \leq T_j \in L_1(-\infty, \infty), \quad \lambda_j \equiv \int_{-\infty}^{\infty} T_j(x)dx \leq 1 \quad j = 1, 2.$$

where E is one of Banach spaces $L_p(-\infty, +\infty) \quad for \quad 1 \leq p \leq \infty$.

2 Operator form of the convolution equation

In this section we will reduce the equation (2) to the operator form and then, we will study on operator form which solvability of this operator form is equvalent to solvabilty of (2).

Here we present the two definition of the following functions

$$f^{\pm}(x) = f(\pm x) \quad ; \quad g^{\pm}(x) = g(\pm x). \tag{3}$$

Considering the relations (3), the equation (2) can be rewritten as

$$f^{+}(x) = g^{+}(x) + \int_{0}^{\infty} T_1(x-t) f^{+}(t)dt + \int_{-\infty}^{0} T_2(x-t) f^{-}(-t)dt, \quad x > 0. \tag{4}$$

In (4), we change $-t$ to t and then the new form of equation(4) will be as

$$f^{+}(x) = g^{+}(x) + \int_{0}^{\infty} T_1(x-t) f^{+}(t)dt + \int_{0}^{\infty} T_2(x+t) f^{-}(t)dt, \quad x > 0. \tag{5}$$

Similarly, we can write

$$f^{-}(-x) = g^{-}(-x) + \int_{0}^{\infty} T_1(-x-t) f^{+}(t)dt + \int_{-\infty}^{0} T_2(-x-t) f^{-}(-t)dt, \quad x < 0 \tag{6}$$

Reformulation of (6) can be considered as

$$f^{-}(x) = g^{-}(x) + \int_{0}^{\infty} T_1(-(x+t)) f^{+}(t)dt + \int_{0}^{\infty} T_2(-(x-t)) f^{-}(t)dt, \quad x > 0. \tag{7}$$

Now the relations (5) and (7) indicate that integral equation (2) is equivalent to the system of integral equations of the form

$$\begin{cases} f^{+}(x) = g^{+}(x) + \int_{0}^{\infty} T_1(x-t) f^{+}(t)dt + \int_{0}^{\infty} T_2(x+t) f^{-}(t)dt, & x > 0 \\ f^{-}(x) = g^{-}(x) + \int_{0}^{\infty} T_1(-(x+t)) f^{+}(t)dt + \int_{0}^{\infty} T_2(-(x-t)) f^{-}(t)dt, & x > 0. \end{cases} \tag{8}$$

Now we define the following functions

$$\begin{aligned} K_1(x) = T_1(x), \quad K_1^0(x) = T_2(x), \qquad x > 0 \\ K_2^0(x) = T_1(-x), \quad K_2(x) = T_1(-x), \qquad x > 0. \end{aligned}$$

By the last notation we can rewrite the system (8) to the following form

$$\begin{cases} f^+(x) = g^+(x) + \int\limits_0^\infty K_1(x-t)f^+(t)dt + \int\limits_0^\infty K_1^0(x+t)f^-(t)dt, & x>0 \\ f^-(x) = g^-(x) + \int\limits_0^\infty K_2^0(x+t)f^+(t)dt + \int\limits_0^\infty K_2(x-t)f^-(t)dt, & x>0 \end{cases} \tag{9}$$

If we define the operators

$$(\hat{K}_1 f)(x) = \int\limits_0^\infty K_1(x-t)f(t)dt,$$

$$(\hat{K}_1^0 f)(x) = \int\limits_0^\infty K_1(x+t)f(t)dt,$$

$$(\hat{K}_2 f)(x) = \int\limits_0^\infty K_2(x-t)f(t)dt,$$

$$(\hat{K}_1^0 f)(x) = \int\limits_0^\infty K_2(x+t)f(t)dt,$$

then, the system (9) will be changed to the equivalent operator form

$$\begin{cases} f^+ = g^+ + \hat{K}_1 f^+ + \hat{K}_1^0 f^-, \\ f^- = g^- + \hat{K}_2^0 f^+ + \hat{K}_2 f^-. \end{cases} \tag{10}$$

System (10) is also equivalent to the system

$$\begin{cases} (\hat{I} - \hat{K}_1)f^+ = g^+ + \hat{K}_1^0 f^-, \\ (\hat{I} - \hat{K}_2)f^- = g^- + \hat{K}_2^0 f^+, \end{cases} \tag{11}$$

where $\hat{I}$ is the identity operator.

3 Factorization of the convolution equation

Let us introduce the following classes of operators

$$\Omega = \{\hat{K} : (\hat{K}\varphi)(x) = \int_0^\infty K(x-t)\varphi(t)dt, \varphi \in L_1(0,\infty), K \in L_1(\mathbb{R})\}$$

$$\Omega^+ = \{\hat{V}^+ : (\hat{V}^+\varphi)(x) = \int_0^x V^+(x-t)\varphi(t)dt, \varphi \in L_1(0,\infty), V^+ \in L_1(0,\infty)\}$$

$$\Omega^- = \{\hat{V}^- : (\hat{V}^-\varphi)(x) = \int_x^\infty V^-(t-x)\varphi(t)dt, \varphi \in L_1(0,\infty), V^- \in L_1(0,\infty)\}$$

$$\Omega^0 = \{\hat{K}^0 : (\hat{V}^0\varphi)(x) = \int_0^\infty K^0(x+t)\varphi(t)dt, \varphi \in L_1(0,\infty), K^+ \in L_1(0,\infty)\} \ .$$

These operators help us to proof the following statements [11]

$$\hat{V}^-\hat{K}^0 \in \Omega^0, \quad \hat{K}^0\hat{V}^+ \in \Omega^0, \quad \hat{V}^-\hat{V}^+ \in \Omega. \tag{12}$$

Now we consider the following factorization

$$(\hat{I} - \hat{K}_j) = (\hat{I} - \hat{V}_j^-)(\hat{I} - \hat{V}_j^+), \tag{13}$$

For $j = 1, 2$, $\hat{K}_j \in \Omega$ are given and $\hat{V}_j^- \in \Omega^-$ and $\hat{V}_j^+ \in \Omega^+$ are unknown operators.
The factorization (13) is equivalent to the Yengibarian's system [2]

$$V^\pm(x) = K^\pm(x) + \int_0^\infty V_j^\mp(t)V_j^\pm(x+t)dt, \quad j = 1, 2. \tag{14}$$

The factorization

$$I - \bar{K}_j(s) = [I - \bar{V}_j^-(s)][I - \bar{V}_j^+(s)], \quad im(s) = 0, \quad j = 1, 2$$

is the symbol of the operators (13). In the above formula $\bar{K}_j$, $\bar{V}_j^+ and$ $\bar{V}_j^-$

are defined as

$$\bar{K}_j(s) = \int\limits_{-\infty}^{\infty} K_j(x)e^{ixs}dx, \quad im(s) = 0, \quad j = 1,2$$

$$\bar{V}_j^+(s) = \int\limits_{-\infty}^{\infty} V_j^+(x)e^{ixs}dx, \quad im(s) \geq 0, \quad j = 1,2$$

$$\bar{V}_j^-(s) = \int\limits_{-\infty}^{\infty} V_j^-(x)e^{ixs}dx, \quad im(s) \leq 0, \quad j = 1,2$$

where $im(s)$ indicates the imaginary part of s.
If we suppose that the left part of (13) is invertible, then the right part of this relation which is contain the Volterra-type operators , will be invertible and their inversion are the renewal-type and have the following form

$$(\hat{I} - \hat{V}^{\pm})^{-1} = \hat{I} + \hat{\Phi}^{\pm}, \quad \hat{\Phi}^{\pm} \in \Omega^{\pm}. \tag{15}$$

In this relation $\Phi^{\pm}$ are defined as

$$\Phi^{\pm}(x) = V^{\pm}(x) + \int\limits_{0}^{x} V^{\pm}(x-t)\Phi^{\pm}(t)dt, \quad x > 0.$$

After using the relations (11) and (13) then we have

$$\begin{cases} (\hat{I} - \hat{V}_1^-)(\hat{I} - \hat{V}_1^+)f^+ = g^+ + \hat{K}_1^0 f^-, \\ \\ (\hat{I} - \hat{V}_2^-)(\hat{I} - \hat{V}_2^+)f^- = g^- + \hat{K}_2^0 f^+. \end{cases} \tag{16}$$

By considering

$$\begin{cases} F^+ = (\hat{I} - \hat{V}_1^+)f^+, \\ \\ F^- = (\hat{I} - \hat{V}_2^+)f^-, \end{cases}$$

relation (16) will become

$$\begin{cases} F^+ = G^+ + \hat{U}_1^0 F^-, \\ \\ F^- = G^- + \hat{U}_2^0 F^+, \end{cases} \tag{17}$$

where

$$\begin{cases} \hat{U}_1^0 = (\hat{I} + \hat{\Phi}_1^-)\hat{K}_1^0(\hat{I} + \hat{\Phi}_2^-), \\ \\ \hat{U}_2^0 = (\hat{I} + \hat{\Phi}_2^-)\hat{K}_2^0(\hat{I} + \hat{\Phi}_1^-), \end{cases} \tag{18}$$

and

$$\begin{cases} G^+ = (\hat{I} - \hat{\Phi}_1^-)g^+, \\ \\ G^- = (\hat{I} - \hat{\Phi}_2^-)g^-. \end{cases} \tag{19}$$

From (12), (19) and (18) it is easy to verify that

$$\hat{U}_{1,2}^0 \in \Omega^0 \quad and \quad G^{\pm} \in E^+.$$

The next theorem shows the conditions in which the equation (17) has a uique solotion .

Theorem 3.1. *If the operators* $\hat{I} - \hat{K}_j, \quad for \quad j = 1, 2$ *are invertible in* E^+ *then*

The system (17)which is contain the Hankel type operators $\hat{U}_{1,2}^0 \in \Omega^0$ *has the unique solution* $(F^+, F^-) \in E^+ \times E^+$. *In this case,* f^+ *and* f^- *can be found by the following system*

$$\begin{cases} f^+ = (\hat{I} + \hat{\Phi}_1^+)F^+ \\ \\ f^- = (\hat{I} + \hat{\Phi}_2^+)F^- \end{cases}$$

proof: See [2].

4 Approximating the solution of convolution equation with two kernels

Numerical approximating of the integral equation of convolution type has worked by a lot of authors ([3], [4] and [6]).In [4], Chandrasekhar introduced the Discrete Ordinate Method (DOM) .
N.B.Yengibaryan(Engibaryan) and E.A.Melkonyan introduced the method for numerical approximating [6],in this method the kernel where is the super position of exponential approximate by following finite linear combination of exponentials

$$K(x) \simeq \tilde{K}(x) = \sum_{m=0}^{N} c_m exp(-s_m x), \quad 0 \leq K(x) \leq \tilde{K}(x),$$

where $s_k \in (a,b) \quad for \quad k = 0,1,2,\cdots,N.$

Pressdorf and et.al., introduced the discrete Wiener-Hopf equations [10, 9], our approach is based on their works.
I changed the integral equation with two kernels (2) to the dual system of of integral equations (9) and showed that this system and hence (2) has answer and now i want to approximate the dual system (9) and by finding its answers, the answers of (2) can be approximate.

remark: Let us remind that a Toeplitz matrix is an $n \times n$ matrix T_n in the form of

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \cdot & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix},$$

where $t_{k,j} = t_{k-j} \quad for \quad k,j = 0,1,2,\cdots,n-1,$ Similarly, a Hankel matrix is an $n \times n$ matrix H_n in the form of

$$H_n = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{(n-1)} \\ h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdot & t_{(n+3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_n & h_{n+1} & \cdots & h_{2n-2} \end{bmatrix},$$

where $h_{k,j} = h_{k+j} \quad for \quad k,j = k,j = 0,1,2,\cdots,n-1.$
We know that there are also the infinite Hankel and Toeplitz matrices of the same figure.
Now let us introduce the Dual algebraic system

$$\begin{cases} f_j^+ = g_j^+ + \sum_{i=1}^{\infty} a_{j+i} f_i^- + \sum_{i=0}^{\infty} a_{j-i} f_i^+ \quad , \quad j = 0,1,2,\cdots \\ \\ f_j^- = g_j^- + \sum_{i=1}^{\infty} b_{-(j+i)} f_i^+ + \sum_{i=0}^{\infty} b_{-(j-i)} f_i^- \quad , \quad j = 1,2,\cdots \end{cases} \tag{20}$$

which is correspond to the dual system of integral equations (8). The same as the integral operators in section (2), here also consider the following matrix operators

$$Kf = \left(\sum_0^{\infty} a_{1-i} f_i, \sum_0^{\infty} a_{2-i} f_i, \sum_0^{\infty} a_{3-i} f_i, \cdots \right)^T$$

$$K^0 f = \sum_{i=1}^{\infty} a_{j+i} f_i \quad , \quad j = 0,1,2,\cdots$$

$$Hf = \sum_{i=0}^{\infty} b_{j-i} f_i \quad , \quad j = 0,1,2,\cdots$$

$$H^0 f = \sum_{i=1}^{\infty} b_{j+i} f_i \quad , \quad j = 0,1,2,\cdots$$

By the above operators, one can change the system (20) to the following form

$$\begin{cases} f^+ = g^+ + Kf^+ + K^0 f^- \\ \\ f^- = g^- + Hf^- + H^0 f^+. \end{cases} \tag{21}$$

We can rewrite (21) as

$$\begin{cases} (I-K)f^+ = g^+ + K^0 f^-, \\ \\ (I-H)f^- = g^- + H^0 f^+. \end{cases} \tag{22}$$

In the previous relation, I is the identity matrix operator. Suppose that the following factorization exists

$$\begin{cases} I-K = (I-K^-)(I-K^+), \\ \\ I-H = (I-H^-)(I-H^+). \end{cases} \tag{23}$$

Let us suppose that the matrices $(I-K^\pm)$ and $(I-H^\pm)$ are invertible and their inversions are in the forms

$$\begin{cases} (I-K^\pm)^{-1} = I + \Phi^\pm, \\ \\ (I-H^\pm)^{-1} = I + \Psi^\pm. \end{cases} \tag{24}$$

From relations (22) and (23) we have

$$\begin{cases} (I-K^-)(I-K^+)f^+ = g^+ + K^0 f^-, \\ \\ (I-H^-)(I-H^+)f^- = g^- + H^0 f^+. \end{cases} \tag{25}$$

Now we define the following vectors

$$\begin{cases} \omega^+ = (I-K^+)f^+, \\ \\ \omega^- = (I-H^+)f^-. \end{cases} \tag{26}$$

From equations (24) and (26) one can write

$$\begin{cases} f^+ = (I + \Phi^+)\omega^+, \\ \\ f^- = (I + \Psi^+)\omega^-. \end{cases} \tag{27}$$

From systems (25), (26) and (27) we can obtain the following system of algebraic equations

$$\begin{cases} (I - K^-)\omega^+ = g^+ + K^0(I + \Psi^+)\omega^-, \\ \\ (I - H^-)\omega^- = g^- + H^0(I + \Phi^+)\omega^+. \end{cases} \tag{28}$$

Multiplying both sides of the first equation in (28) by $(I - K^-)^{-1}$ and both sides of the second equation by $(I - H^-)^{-1}$, then we can receive to the following system

$$\begin{cases} \omega^+ = (I + \Phi^-)g^+ + (I + \Phi^-)K^0(I + \Psi^+)\omega^-, \\ \\ \omega^- = (I + \Psi^-)g^- + (I + \Psi^-)H^0(I + \Phi^+)\omega^+. \end{cases} \tag{29}$$

Now we introduce the following vectors

$$\begin{aligned} (I + \Phi^-)g^+ &= G^+, \\ (I + \Psi^-)g^- &= G^-, \\ (I + \Phi^-)K^0(I + \Psi^+) &= U_1^0, \\ (I + \Psi^-)H^0(I + \Phi^+) &= U_2^0. \end{aligned}$$

These four vectors help us to write the system (29) as

$$\begin{cases} \omega^+ = G^+ + U_1^0\omega^-, \\ \\ \omega^- = G^- + U_2^0\omega^+. \end{cases} \tag{30}$$

If we take into account the properties of the classes $\Omega, \Omega^\pm$ and Ω^0, then we can see that

$$U_{1,2}^0 \in \Omega^0, \quad G^\pm \in E^+. \tag{31}$$

The factorization of matrix operators (23) are equivalent to the corresponding nonlinear systems

$$\begin{cases} K_i^+ = a_i + \sum\limits_{j=0}^{\infty} K_j^- K_{i+j}^+, \quad i = 0, 1, 2, \cdots \\ \\ K_i^- = a_{-i} + \sum\limits_{j=0}^{\infty} K_{i+j}^- K_j^+, \quad i = 0, 1, 2, \cdots . \end{cases} \tag{32}$$

Finally, the factorization of matrix operators (23) reduce the solution of system (20) to the solution of the following systems of equations with triangular matrices

$$\begin{cases} \rho_i^+ - \sum\limits_{j=i}^{\infty} K_{j-i}^- \rho_j^+ = \eta_i^+ + (K^0 f^-)_i, \quad i = 0, 1, 2, \cdots \\ \\ f_i^+ - \sum\limits_{j=0}^{i} K_{i-j}^- f_j^- = \rho_i^+, \qquad i = 0, 1, 2, \cdots \end{cases} \tag{33}$$

and

$$\begin{cases} \theta_i^+ - \sum\limits_{j=i}^{\infty} H_{j-i}^- \theta_j^+ = \eta_i^- + (H^0 f^+)_i, \quad i = 0, 1, 2, \cdots \\ \\ f_i^- - \sum\limits_{j=0}^{i} H_{i-j}^+ f_j^- = \theta_i^+, \qquad i = 0, 1, 2, \cdots \end{cases} \tag{34}$$

Here we propose the following theorem.

Theorem 4.1. *The dual system (20) can be solved by the following scheme:*

1. *We solve (32) using successive iterative method.*
   (For more exploration about successive iterative method one can refer to "Integral Equation" books)

2. *From (33) and (34) we can compute* $K^{\pm}$ *and* $H^{\pm}$. *It makes it possible to also find the* ω^{+} *and* ω^{-} *from (30)*

3. f^{+} *and* f^{-} *can be obtained from (27).*

4. *Dual integral equation (9) can be approximated from last step.*

5. *solution of equation (2) can be approximated.*

5 Conclusions

We introduced and studied a new integral equation of convolution type with two kernels, and after some manipulations we showed that it is solvable. In the latest section we introduced a new way for approximating of this kind of equations.

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