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Convolution Integral Equations with Two Kernels

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Abstract. In this paper, we consider the integral equation of convolution type with some conditions which has two kernels. At first we change this integral equation to a simple operator form and then, we approximate it by Topelitzian and Hankelian series.

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1. Introduction

Integral equation on the half-line with a kernel that depends on the difference of the arguments are called Wiener-Hopf Integral Equations. In this paper, we use the notation WHIE for this type of integral equations. This type of integral equations have important applications in different areas of mathematical and natural sciences such as neutron transport theory, nuclear reactor, astrophysics and atmosphere optic [1, 3, 4]. A new factorization method recently found by Yengibaryan [2] leads in many cases to analytical or effective numerical solution of the WHIE where the integral equation of the convolution type has the following

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form

$$f(x) = g(x) + \int_{-\infty}^{\infty} k(x-t)f(t)dt.$$

Ambartsumian's invariance principle led to a wide and useful application in the theory of WHIE of nonlinear functional equations Ambartsumian's equation which is in the following form

$$\varphi(s) = 1 + \varphi(s) \int_{a}^{b} \frac{1}{s+p} \varphi(p) d\sigma(p), \tag{1}$$

plays an important role in effective solution and the mathematical study of special classes of the nonsingular and singular WHIE [1, 4].

This paper is devoted to investigation on the integral equation of the convolution type with two kernels on the entire line in the following form

$$f(x) = g(x) + \int_{0}^{\infty} T_1(x-t)f(t)dt + \int_{-\infty}^{0} T_2(x-t)f(t)dt, \quad x \in \mathbb{R} \tag{2}$$

where $f \in L_1^{loc}(-\infty, \infty)$ is the unknown function and we know a function is called locally integrable if, around every point in the domain, there is a neighborhood on which the function is integrable and $g, T_{1,2}$ are the functions that have the following properties

$$g \in E, \quad 0 \leq T_j \in L_1(-\infty, \infty), \quad \lambda_j \equiv \int_{-\infty}^{\infty} T_j(x)dx \leq 1 \quad j = 1, 2,$$

where E is one of Banach spaces $L_p(-\infty, +\infty) \quad for \quad 1 \leq p \leq \infty$.

2. Operator Form of the Convolution Equation

In this section we will reduce equation (2) to the operator form and then, we will study in operator form solvability of which of these operator forms is equivalent to solvabilty of (2).

Here we present the definitions of the following functions

$$f^{\pm}(x) = f(\pm x) \quad ; \quad g^{\pm}(x) = g(\pm x). \tag{3}$$

Considering relations (3), equation (2) can be rewritten as

$$f^{+}(x) = g^{+}(x) + \int_{0}^{\infty} T_1(x-t)f^{+}(t)dt + \int_{-\infty}^{0} T_2(x-t)f^{-}(-t)dt, \quad x > 0. \tag{4}$$

In (4), we change $-t$ to t and then the new form of equation(4) will be obtained as

$$f^{+}(x) = g^{+}(x) + \int_{0}^{\infty} T_1(x-t)f^{+}(t)dt + \int_{0}^{\infty} T_2(x+t)f^{-}(t)dt, \quad x > 0. \tag{5}$$

Similarly, we can write

$$f^{-}(-x) = g^{-}(-x) + \int_{0}^{\infty} T_1(-x-t)f^{+}(t)dt + \int_{-\infty}^{0} T_2(-x-t)f^{-}(-t)dt, \quad x < 0 \tag{6}$$

Reformulation of (6) can be considered as

$$f^{-}(x) = g^{-}(x) + \int_{0}^{\infty} T_1(-(x+t))f^{+}(t)dt + \int_{0}^{\infty} T_2(-(x-t))f^{-}(t)dt, \quad x > 0. \tag{7}$$

Relations (5) and (7) indicate that integral equation (2) is equivalent to the system of integral equations of the form

$$\begin{cases} f^{+}(x) = g^{+}(x) + \int_{0}^{\infty} T_1(x-t)f^{+}(t)dt + \int_{0}^{\infty} T_2(x+t)f^{-}(t)dt, & x > 0 \\ f^{-}(x) = g^{-}(x) + \int_{0}^{\infty} T_1(-(x+t))f^{+}(t)dt + \int_{0}^{\infty} T_2(-(x-t))f^{-}(t)dt, & x > 0. \end{cases} \tag{8}$$

Now we define the following functions

$$\begin{aligned} K_1(x) = T_1(x), \quad K_1^0(x) = T_2(x), \qquad x > 0 \\ K_2^0(x) = T_1(-x), \quad K_2(x) = T_1(-x), \qquad x > 0. \end{aligned}$$

By the last notation we can rewrite the system (8) to the following form

$$\begin{cases} f^{+}(x)=g^{+}(x)+\int\limits_{0}^{\infty}K_1(x-t)f^{+}(t)dt+\int\limits_{0}^{\infty}K_1^0(x+t)f^{-}(t)dt, & x>0 \\ f^{-}(x)=g^{-}(x)+\int\limits_{0}^{\infty}K_2^0(x+t)f^{+}(t)dt+\int\limits_{0}^{\infty}K_2(x-t)f^{-}(t)dt, & x>0 \end{cases} \tag{9}$$

If we define the operators

$$(\hat{K}_1 f)(x)=\int\limits_{0}^{\infty}K_1(x-t)f(t)dt,$$

$$(\hat{K}_1^0 f)(x)=\int\limits_{0}^{\infty}K_1(x+t)f(t)dt,$$

$$(\hat{K}_2 f)(x)=\int\limits_{0}^{\infty}K_2(x-t)f(t)dt,$$

$$(\hat{K}_1^0 f)(x)=\int\limits_{0}^{\infty}K_2(x+t)f(t)dt,$$

then, the system (10) will be changed to the equivalent operator form

$$\begin{cases} f^{+}=g^{+}+\hat{K}_1 f^{+}+\hat{K}_1^0 f^{-}, \\ f^{-}=g^{-}+\hat{K}_2^0 f^{+}+\hat{K}_2 f^{-}. \end{cases} \tag{10}$$

System (10) is also equivalent to the system

$$\begin{cases} (\hat{I}-\hat{K}_1)f^{+}=g^{+}+\hat{K}_1^0 f^{-}, \\ (\hat{I}-\hat{K}_2)f^{-}=g^{-}+\hat{K}_2^0 f^{+}, \end{cases} \tag{11}$$

where $\hat{I}$ is the identity operator.

3. Factorization of the Convolution Equation

Let us introduce the following classes of operators

$$\Omega = \{\hat{K} : (\hat{K}\varphi)(x) = \int_0^\infty K(x-t)\varphi(t)dt, \varphi \in L_1(0,\infty), K \in L_1(\mathbb{R})\}$$

$$\Omega^+ = \{\hat{V}^+ : (\hat{V}^+\varphi)(x) = \int_0^x V^+(x-t)\varphi(t)dt, \varphi \in L_1(0,\infty), V^+ \in L_1(0,\infty)\}$$

$$\Omega^- = \{\hat{V}^- : (\hat{V}^-\varphi)(x) = \int_x^\infty V^-(t-x)\varphi(t)dt, \varphi \in L_1(0,\infty), V^- \in L_1(0,\infty)\}$$

$$\Omega^0 = \{\hat{K}^0 : (\hat{V}^0\varphi)(x) = \int_0^\infty K^0(x+t)\varphi(t)dt, \varphi \in L_1(0,\infty), K^+ \in L_1(0,\infty)\} \ .$$

These operators help us to prove the following statements [11]

$$\hat{V}^-\hat{K}^0 \in \Omega^0, \quad \hat{K}^0\hat{V}^+ \in \Omega^0, \quad \hat{V}^-\hat{V}^+ \in \Omega. \tag{12}$$

Now we consider the following factorization

$$(\hat{I} - \hat{K}_j) = (\hat{I} - \hat{V}_j^-)(\hat{I} - \hat{V}_j^+), \tag{13}$$

For $j = 1, 2$, $\hat{K}_j \in \Omega$ are given and $\hat{V}_j^- \in \Omega^-$ and $\hat{V}_j^+ \in \Omega^+$ are unknown operators.
The factorization (13) is equivalent to the Yengibarian's system [2]

$$V^\pm(x) = K^\pm(x) + \int_0^\infty V_j^\mp(t)V_j^\pm(x+t)dt, \quad j = 1, 2. \tag{14}$$

The factorization

$$I - \bar{K}_j(s) = [I - \bar{V}_j^-(s)][I - \bar{V}_j^+(s)], \quad im(s) = 0, \quad j = 1, 2$$

is the symbol of the operators (13). In the above formula $\bar{K}_j, \bar{V}_j^+ and \quad \bar{V}_j^-$

are defined as

$$\bar{K}_j(s) = \int_{-\infty}^{\infty} K_j(x)e^{ixs}dx, \quad im(s) = 0, \quad j = 1, 2$$

$$\bar{V}_j^+(s) = \int_{-\infty}^{\infty} V_j^+(x)e^{ixs}dx, \quad im(s) \geq 0, \quad j = 1, 2$$

$$\bar{V}_j^-(s) = \int_{-\infty}^{\infty} V_j^-(x)e^{ixs}dx, \quad im(s) \leq 0, \quad j = 1, 2$$

where $im(s)$ indicates the imaginary part of s.
If we suppose that the operator in the left part of (13) is invertible, then the right part of this relation which is of Volterra-type, will be invertible and their inversion are the renewal-type and have the following form

$$(\hat{I} - \hat{V}^{\pm})^{-1} = \hat{I} + \hat{\Phi}^{\pm}, \quad \hat{\Phi}^{\pm} \in \Omega^{\pm}. \tag{15}$$

In this relation $\Phi^{\pm}$ are defined as

$$\Phi^{\pm}(x) = V^{\pm}(x) + \int_0^x V^{\pm}(x-t)\Phi^{\pm}(t)dt, \quad x > 0.$$

After using the relations (11) and (13) then we have

$$\begin{cases} (\hat{I} - \hat{V}_1^-)(\hat{I} - \hat{V}_1^+)f^+ = g^+ + \hat{K}_1^0 f^-, \\ (\hat{I} - \hat{V}_2^-)(\hat{I} - \hat{V}_2^+)f^- = g^- + \hat{K}_2^0 f^+. \end{cases} \tag{16}$$

By considering

$$\begin{cases} F^+ = (\hat{I} - \hat{V}_1^+)f^+, \\ F^- = (\hat{I} - \hat{V}_2^+)f^-, \end{cases}$$

relation (16) will become

$$\begin{cases} F^+ = G^+ + \hat{U}_1^0 F^-, \\ F^- = G^- + \hat{U}_2^0 F^+, \end{cases} \tag{17}$$

where

$$\begin{cases} \hat{U}_1^0 = (\hat{I} + \hat{\Phi}_1^-)\hat{K}_1^0(\hat{I} + \hat{\Phi}_2^-), \\ \\ \hat{U}_2^0 = (\hat{I} + \hat{\Phi}_2^-)\hat{K}_2^0(\hat{I} + \hat{\Phi}_1^-), \end{cases} \tag{18}$$

and

$$\begin{cases} G^+ = (\hat{I} - \hat{\Phi}_1^-)g^+, \\ \\ G^- = (\hat{I} - \hat{\Phi}_2^-)g^-. \end{cases} \tag{19}$$

From (12), (19) and (18) it is easy to verify that

$$\hat{U}_{1,2}^0 \in \Omega^0 \quad and \quad G^{\pm} \in E^+.$$

The next theorem shows the conditions under which equation (17) has a uique solotion .

Theorem 3.1. *If operators $\hat{I} - \hat{K}_j, \quad for \quad j = 1, 2$ are invertible 0n E^+ then the system (17)which contains the Hankel type operators $\hat{U}_{1,2}^0 \in \Omega^0$ has the unique solution $(F^+, F^-) \in E^+ \times E^+$. In this case, f^+ and f^- can be found by solving the following system*

$$\begin{cases} f^+ = (\hat{I} + \hat{\Phi}_1^+)F^+ \\ \\ f^- = (\hat{I} + \hat{\Phi}_2^+)F^- \end{cases}$$

proof: See [2].

4. Approximating the Solution of Convolution Equation with Two Kernels

Numerical approximation of integral equations of convolution type has been considered by many authors ([3], [4] and [6]).In [4], Chandrasekhar introduced the Discrete Ordinate Method (DOM) .

N.B.Yengibaryan(Engibaryan) and E.A.Melkonyan introduced the method for numerical approximation [6]. In this method the kernel is the superposition of exponential approximates by following finite linear combination of exponentials

$$K(x) \simeq \tilde{K}(x) = \sum_{m=0}^{N} c_m exp(-s_m x), \quad 0 \leq K(x) \leq \tilde{K}(x),$$

where $s_k \in (a,b) \quad for \quad k = 0,1,2,\cdots,N.$

Pressdorf et.al., introduced the discrete Wiener-Hopf equations [10, 9]. Our approach is based on their works.
We have changed the integral equation with two kernels (2) to the dual system of integral equations (10) and showed that this system and hence (2) has solution and now we are going to approximate the dual system (10) and by finding its solutions, the solution of (2) can be approximated.

Remark 4.1. *Let us remind that a Toeplitz matrix is an $n \times n$ matrix T_n in the form*

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \cdot & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix},$$

where $t_{k,j} = t_{k-j} \quad for \quad k,j = 0,1,2,\cdots,n-1,$ Similarly, a Hankel matrix is an $n \times n$ matrix H_n in the form

$$H_n = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{(n-1)} \\ h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdot & t_{(n+3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_n & h_{n+1} & \cdots & h_{2n-2} \end{bmatrix},$$

where $h_{k,j} = h_{k+j} \quad for \quad k,j = k,j = 0,1,2,\cdots,n-1.$

We know that there are also the infinite dimensional Hankel and Toeplitz matrices of the same type.
Now let us introduce the Dual algebraic system

$$\begin{cases} f_j^+ = g_j^+ + \sum\limits_{i=1}^{\infty} a_{j+i} f_i^- + \sum\limits_{i=0}^{\infty} a_{j-i} f_i^+ \quad , \quad j = 0, 1, 2, \dots, \\ \\ f_j^- = g_j^- + \sum\limits_{i=1}^{\infty} b_{-(j+i)} f_i^+ + \sum\limits_{i=0}^{\infty} b_{-(j-i)} f_i^- \quad , \quad j = 1, 2, \dots, \end{cases} \tag{20}$$

which corresponds to the dual system of integral equations (8).
Like the integral operators in section (2), here also we consider the following matrix operators

$$Kf = \left(\sum_0^{\infty} a_{1-i} f_i, \sum_0^{\infty} a_{2-i} f_i, \sum_0^{\infty} a_{3-i} f_i, \dots, \right)^T$$

$$K^0 f = \sum_{i=1}^{\infty} a_{j+i} f_i \quad , \quad j = 0, 1, 2, \dots,$$

$$Hf = \sum_{i=0}^{\infty} b_{j-i} f_i \quad , \quad j = 0, 1, 2, \dots,$$

$$H^0 f = \sum_{i=1}^{\infty} b_{j+i} f_i \quad , \quad j = 0, 1, 2, \dots.$$

By the above operators, one can change system (20) into the following form

$$\begin{cases} f^+ = g^+ + Kf^+ + K^0 f^- \\ \\ f^- = g^- + Hf^- + H^0 f^+. \end{cases} \tag{21}$$

Also we can rewrite (21) as

$$\begin{cases} (I - K) f^+ = g^+ + K^0 f^-, \\ \\ (I - H) f^- = g^- + H^0 f^+. \end{cases} \tag{22}$$

In the previous relation, I is the identity operator.
Suppose that the following factorizations exist

$$\begin{cases} I - K = (I - K^-)(I - K^+), \\ \\ I - H = (I - H^-)(I - H^+). \end{cases} \tag{23}$$

Let us suppose that the matrices $(I - K^{\pm})$ and $(I - H^{\pm})$ are invertible and their inverses are in the forms

$$\begin{cases} (I - K^{\pm})^{-1} = I + \Phi^{\pm}, \\ \\ (I - H^{\pm})^{-1} = I + \Psi^{\pm}. \end{cases} \tag{24}$$

From relations (22) and (23) we have

$$\begin{cases} (I - K^-)(I - K^+)f^+ = g^+ + K^0 f^-, \\ \\ (I - H^-)(I - H^+)f^- = g^- + H^0 f^+. \end{cases} \tag{25}$$

Now we define the following vectors

$$\begin{cases} \omega^+ = (I - K^+)f^+, \\ \\ \omega^- = (I - H^+)f^-. \end{cases} \tag{26}$$

From equations (24) and (26) one can write

$$\begin{cases} f^+ = (I + \Phi^+)\omega^+, \\ \\ f^- = (I + \Psi^+)\omega^-. \end{cases} \tag{27}$$

Moreover from systems (25), (26) and (27) we can obtain the following system of algebraic equations

$$\begin{cases} (I - K^-)\omega^+ = g^+ + K^0(I + \Psi^+)\omega^-, \\ \\ (I - H^-)\omega^- = g^- + H^0(I + \Phi^+)\omega^+. \end{cases} \tag{28}$$

Multiplying both sides of the first equation in (28) by $(I-K^-)^{-1}$ and both sides of the second equation by $(I-H^-)^{-1}$, we obtain the following system

$$\begin{cases} \omega^+ = (I+\Phi^-)g^+ + (I+\Phi^-)K^0(I+\Psi^+)\omega^-, \\ \\ \omega^- = (I+\Psi^-)g^- + (I+\Psi^-)H^0(I+\Phi^+)\omega^+. \end{cases} \tag{29}$$

Now we introduce the following vectors

$$\begin{aligned} (I+\Phi^-)g^+ &= G^+, \\ (I+\Psi^-)g^- &= G^-, \\ (I+\Phi^-)K^0(I+\Psi^+) &= U_1^0, \\ (I+\Psi^-)H^0(I+\Phi^+) &= U_2^0. \end{aligned}$$

These four vectors help us to write the system (29) as

$$\begin{cases} \omega^+ = G^+ + U_1^0\omega^-, \\ \\ \omega^- = G^- + U_2^0\omega^+. \end{cases} \tag{30}$$

If we take into account the properties of the classes $\Omega, \Omega^\pm$ and Ω^0, then we can see that

$$U_{1,2}^0 \in \Omega^0, \quad G^\pm \in E^+. \tag{31}$$

By factorization the matrix operators (23) we obtain the corresponding nonlinear systems

$$\begin{cases} K_i^+ = a_i + \sum\limits_{j=0}^{\infty} K_j^- K_{i+j}^+, \quad i = 0, 1, 2, \cdots \\ \\ K_i^- = a_{-i} + \sum\limits_{j=0}^{\infty} K_{i+j}^- K_j^+, \quad i = 0, 1, 2, \cdots . \end{cases} \tag{32}$$

Finally, factorization of matrix operators (23) reduces the solution of system (20) to the solution of the following systems of equations with triangular matrices

$$\begin{cases} \rho_i^+ - \sum_{j=i}^{\infty} K_{j-i}^- \rho_j^+ = \eta_i^+ + (K^0 f^-)_i, & i = 0, 1, 2, \cdots \\ \\ f_i^+ - \sum_{j=0}^{i} K_{i-j}^- f_j^- = \rho_i^+, & i = 0, 1, 2, \cdots \end{cases} \tag{33}$$

and

$$\begin{cases} \theta_i^+ - \sum_{j=i}^{\infty} H_{j-i}^- \theta_j^+ = \eta_i^- + (H^0 f^+)_i, & i = 0, 1, 2, \cdots \\ \\ f_i^- - \sum_{j=0}^{i} H_{i-j}^+ f_j^- = \theta_i^+, & i = 0, 1, 2, \cdots \end{cases} \tag{34}$$

Here we propose the following theorem.

Theorem 4.2. *The dual system (20) can be solved by the following scheme:*

1. *We solve (32) using successive iterative method.*
   (For more exploration about successive iterative method one can refer to "Integral Equation" books, for example "Linear and Nonlinear Integral Equations Methods and Applications" was written by "Abdul-Majid Wazwaz")
2. *From (33) and (34) we can compute $K^{\pm}$ and $H^{\pm}$. Also it is possible to find ω^+ and ω^- from (30)*
3. *f^+ and f^- can be obtained from (27).*
4. *Dual integral equation (10) can be approximated from the last step.*
5. *Solution of equation (2) can be approximated.*

5. Conclusions

We introduced and studied a new integral equation of convolution type with two kernels, and after some manipulations we showed that it is solvable. In the last section we introduced a new way for approximating the solutions of this kind of equations.

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