Journal of Mathematical Extension
Vol. 10, No. 2, (2016), 59-73
ISSN: 1735-8299
URL: http://www.ijmex.com

Some Results on the Growth Analysis of Entire Functions Using their Maximum Terms and Relative L^*-orders

S. K. Datta*
University of Kalyani

T. Biswas
Rajbari Rabindrapalli

A. Hoque
University of Kalyani

Abstract. In this paper we study some comparative growth properties of composite entire functions in terms of their maximum terms on the basis of their relative L^* order (relative L^* lower order) with respect to another entire function.

AMS Subject Classification: 30D20; 30D30; 30D35
Keywords and Phrases: Entire function, maximum term, composition, relative L^* order (relative L^* lower order), growth.

1. Introduction

Let $\mathbb{C}$ be the set of all finite complex numbers. Also let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu_f(r)$ and the maximum modulus $M_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are defined as $\mu_f(r) = \max(|a_n| r^n)$ and $M_f(r) = \max_{|z|=r} |f(z)|$ respectively. We use

Received: July 2015; Accepted: February 2016
*Corresponding author

the standard notations and definitions in the theory of entire functions which are available in [10]. In the sequel we use the following notation:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right), k = 1, 2, 3, ... and \ \log^{[0]} x = x.$$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. Bernal [1] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

Similarly, one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

If we consider $g(z) = \exp z$, the above definition coincides with the classical definition [9] of order (lower order) of an entire function f which is as follows:

Definition 1.1. *The order ρ_f and the lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \ \ and \ \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

Using the inequalities $\mu_f(r) \leqslant M_f(r) \leqslant \frac{R}{R-r}\mu_f(R)$ for $0 \leqslant r < R$ [8] one may give an alternative definition of entire function in the following manner:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r}.$$

Now let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker [5] defined it in the following way:

Definition 1.2. [5] *A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon(>0)$,*

$$\frac{1}{k^{\varepsilon}} \leqslant \frac{L(kr)}{L(r)} \leqslant k^{\varepsilon} \ for \ r \geqslant r(\varepsilon)$$

and uniformly for $k(\geqslant 1)$.
If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0.$$

Somasundaram and Thamizharasi [6] introduced the notions of L-order for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order for entire function is L^*-order and its definition is as follows:

Definition 1.3. [6] *The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \ and \ \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

In view of the inequalities $\mu_f(r) \leqslant M_f(r) \leqslant \frac{R}{R-r}\mu_f(R)$ for $0 \leqslant r < R$ [8] one may verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]}.$$

In the line of Somasundaram and Thamizharasi [6] and Bernal [1], Datta and Biswas [2] gave the definition of relative L^*-order of an entire function in the following way:

Definition 1.4. [2] *The relative L^*-order of an entire function f with respect to another entire function g , denoted by $\rho_g^{L^*}(f)$ in the following way*

$$\begin{aligned}\rho_g^{L^*}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left\{re^{L(r)}\right\}^{\mu} \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log\left[re^{L(r)}\right]}.\end{aligned}$$

Similarly, one can define the relative L^*-lower order of f with respect to g denoted by $\lambda_g^{L^*}(f)$ as follows:

$$\lambda_g^{L^*}(f) = \liminf_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log\left[re^{L(r)}\right]}.$$

In the case of relative L^*-order (relative L^*-lower order), it therefore seems reasonable to define suitably an alternative definition of relative L^*-order (relative L^*-lower order) of entire function in terms of its maximum terms. Datta, Biswas and Ali [4] also introduced such definition in the following way:

Definition 1.5. [4] *The relative order $\rho_g^{L^*}(f)$ and the relative lower order $\lambda_g(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty}\frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty}\frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]}.$$

In this paper we wish to establish some results relating to the growth rates of composite entire functions in terms of their maximum terms on the basis of relative L^*-order (relative L^*-lower order).

2. Main Results

In the following we present some lemmas which will be needed in the sequel.

Lemma 2.1. [7] *Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leqslant \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R - r}\mu_g(R)\right).$$

Lemma 2.2. [7] *If f and g are any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geqslant \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right).$$

Lemma 2.3. [3] *If f be an entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geqslant \beta \mu_f(r).$$

Theorem 2.4. *Let f, g and h be any three entire functions and $g(0) = 0$. If there exist α and β, satisfying $0 < \alpha < 1$, $\beta > 0$ and $\alpha(\beta + 1) > 1$, such that*

$$\begin{aligned}(i)\ \limsup_{r\to\infty}\frac{\log \mu_h^{-1}(\mu_g(r))}{\left(\log re^{L(r)}\right)^{\alpha}} &= A, \text{ a real number } > 0,\\ (ii)\ \liminf_{r\to\infty}\frac{\log \mu_h^{-1}(\mu_f(r))}{\left(\log M_h^{-1}(r)\right)^{\beta+1}} &= B, \text{ a real number } > 0.\end{aligned}$$

Then

$$\rho_h^{L^*}(f\circ g) = \infty.$$

Proof. From (i), we have for a sequence of values of r tending to infinity

$$\log \mu_h^{-1}(\mu_g(r)) \geqslant (A - \varepsilon)\left(\log re^{L(r)}\right)^{\alpha} \tag{1}$$

and from (ii), we obtain for all sufficiently large values of r that

$$\log \mu_h^{-1}(\mu_f(r)) \geqslant (B - \varepsilon)\left(\log \mu_h^{-1}(r)\right)^{\beta+1}.$$

Since $\mu_g(r)$ is continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\log \mu_h^{-1}\left(\mu_f(\mu_g(r))\right) \geqslant (B-\varepsilon)\left(\log \mu_h^{-1}\left(\mu_g(r)\right)\right)^{\beta+1} . \qquad (2)$$

Also $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2, Lemma 2.3, (1) and (2) for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log \mu_h^{-1}\mu_{f\circ g}(r) &\geqslant \log \mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{2}\right)\right)\right\} \\
i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) &\geqslant \log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\} \\
i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) &\geqslant (B-\varepsilon)\left(\log \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right)^{\beta+1} \\
i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) &\geqslant (B-\varepsilon)\left[(A-\varepsilon)\left(\log\left(\frac{r}{100}\right)e^{L\left(\frac{r}{100}\right)}\right)^{\alpha}\right]^{\beta+1} \\
i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) &\geqslant (B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left(\frac{r}{100}\right)e^{L\left(\frac{r}{100}\right)}\right)^{\alpha(\beta+1)} \\
i.e.,\ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} &\geqslant \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left[\log\left(\frac{r}{100}\right)e^{L\left(\frac{r}{100}\right)}\right]^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]}
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ \limsup_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \\
&\geqslant \liminf_{r\to\infty}\frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left[\log re^{L(r)}+O(1)\right]^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]} .
\end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha(\beta+1)>1$, it follows from above that

$$\rho_h^{L^*}(f\circ g)=\infty,$$

which proves the theorem. □

In the line of Theorem 2.4, one may state the following two theorems without their proofs :

Theorem 2.5. *Let f, g and h be any three entire functions and $g(0) = 0$. If there exist α and β, satisfying $0 < \alpha < 1$, $\beta > 0$ and $\alpha(\beta+1) > 1$, such that*

$$(i) \liminf_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_g(r))}{\left(\log re^{L(r)}\right)^{\alpha}} = A, \text{ a real number } > 0,$$
$$(ii) \limsup_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_f(r))}{\left(\log \mu_h^{-1}(r)\right)^{\beta+1}} = B, \text{ a real number } > 0.$$

Then

$$\rho_h^{L^*}(f\circ g) = \infty.$$

Theorem 2.6. *Let f, g and h be any three entire functions and $g(0) = 0$. If there exist α and β, satisfying $0 < \alpha < 1$, $\beta > 0$ and $\alpha(\beta+1) > 1$, such that*

$$(i) \liminf_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_g(r))}{\left(\log re^{L(r)}\right)^{\alpha}} = A, \text{ a real number } > 0,$$
$$(ii) \liminf_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_f(r))}{\left(\log \mu_h^{-1}(r)\right)^{\beta+1}} = B, \text{ a real number } > 0.$$

Then

$$\lambda_h^{L^*}(f\circ g) = \infty.$$

Theorem 2.7. *Let f, g and h be any three entire functions and $g(0) = 0$. If there exist α and β, satisfying $\alpha > 1$, $0 < \beta < 1$ and $\alpha\beta > 1$, such that*

$$(i) \limsup_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_g(r))}{\left(\log^{[2]} r\right)^{\alpha}} = A, \text{ a real number } > 0,$$
$$(ii) \liminf_{r\to\infty} \frac{\log\left[\frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)}\right]}{\left[\log \mu_h^{-1}(r)\right]^{\beta}} = B, \text{ a real number } > 0.$$

Then

$$\rho_h^{L^*}(f \circ g) = \infty.$$

Proof. From (i), we have for a sequence of values of r tending to infinity we get that

$$\log \mu_h^{-1}(\mu_g(r)) \geqslant (A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha} \tag{3}$$

and from (ii), we obtain for all sufficiently large values of r that

$$\begin{aligned} \log\left[\frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)}\right] &\geqslant (B-\varepsilon)\left[\log \mu_h^{-1}(r)\right]^{\beta} \\ i.e., \ \frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)} &\geqslant \exp\left[(B-\varepsilon)\left[\log \mu_h^{-1}(r)\right]^{\beta}\right]. \end{aligned}$$

Since $\mu_g(r)$ is continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\frac{\log \mu_h^{-1}(\mu_f(\mu_g(r)))}{\log \mu_h^{-1}(\mu_g(r))} \geqslant \exp\left[(B-\varepsilon)\left[\log \mu_h^{-1}(\mu_g(r))\right]^{\beta}\right]. \tag{4}$$

Also $\mu_h^{-1}(r)$ is increasing function of r, it follows from Lemma 2.2, Lemma 2.3, (3) and (4) for a sequence of values of r tending to infinity that

$$\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geqslant \frac{\log \mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}}{\log\left[re^{L(r)}\right]},$$

$$i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geqslant \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log\left[re^{L(r)}\right]},$$

$$\begin{aligned} & i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \\ \geqslant \ & \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log\left[re^{L(r)}\right]}, \end{aligned}$$

$$i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\geqslant \ \exp\left[(B-\varepsilon)\left[\log \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right]^{\beta}\right]\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\geqslant \ \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta}\right]\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\geqslant \ \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}\log^{[2]}\left(\frac{r}{100}\right)\right]\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\geqslant \ \left(\log\left(\frac{r}{100}\right)\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}}\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]}$$

$$i.e., \ \limsup_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]},$$

$$\geqslant \ \liminf_{r\to\infty}\left(\log\left(\frac{r}{100}\right)\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}}\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha>1$, $\alpha\beta>1$, the theorem follows from above. □

In the line of Theorem 2.7, one may also state the following two theorems without their proofs :

Theorem 2.8. *Let f, g and h be any three entire functions and $g(0)=0$. If there exist α and β, satisfying $\alpha>1,\ 0<\beta<1$ and $\alpha\beta>1$, such* *that*

$$(i) \liminf_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_g(r))}{\left(\log^{[2]} r\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii) \limsup_{r\to\infty} \frac{\log\left[\frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)}\right]}{\left[\log \mu_h^{-1}(r)\right]^{\beta}} = B, \text{ a real number } > 0.$$

Then

$$\rho_h^{L^*}(f \circ g) = \infty.$$

Theorem 2.9. *Let f, g and h be any three entire functions and $g(0) = 0$. If there exist α and β, satisfying $\alpha > 1,\ 0 < \beta < 1$ and $\alpha\beta > 1$, such that*

$$(i) \liminf_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_g(r))}{\left(\log^{[2]} r\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii) \liminf_{r\to\infty} \frac{\log\left[\frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)}\right]}{\left[\log \mu_h^{-1}(r)\right]^{\beta}} = B, \text{ a real number } > 0.$$

Then

$$\lambda_h^{L^*}(f \circ g) = \infty.$$

Theorem 2.10. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(g) \leqslant \rho_h^{L^*}(g) < \infty$, $g(0) = 0$ and*

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)} = A, \text{ a real number } < \infty.$$

Then

$$\lambda_h^{L^*}(f \circ g) \leqslant A \cdot \lambda_h^{L^*}(g) \text{ and } \rho_h^{L^*}(f \circ g) \leqslant A \cdot \rho_h^{L^*}(g).$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2 for all sufficiently large values of r that

$$\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \leqslant \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log\left[re^{L(r)}\right]},$$

$$i.e.,\ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \leqslant \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)} \cdot \frac{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log\left[re^{L(r)}\right]}, \tag{5}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \leqslant \liminf_{r\to\infty}\left[\frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)} \cdot \frac{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log\left[re^{L(r)}\right]}\right],$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \leqslant \limsup_{r\to\infty}\frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)} \cdot \liminf_{r\to\infty}\frac{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log\left[re^{L(r)}\right]},$$

$$i.e.,\ \lambda_h^{L^*}\left(f\circ g\right) \leqslant A\cdot \lambda_h^{L^*}\left(g\right)\ . \tag{6}$$

Also from (5), we obtain for all sufficiently large values of r that

$$\limsup_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \leqslant \limsup_{r\to\infty}\left[\frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)} \cdot \frac{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log\left[re^{L(r)}\right]}\right]$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \leqslant \limsup_{r\to\infty}\frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)} \cdot \limsup_{r\to\infty}\frac{\log \mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \rho_h^{L^*}(fog) \leqslant A \cdot \rho_h^{L^*}(g). \tag{7}$$

Therefore the theorem follows from (6) and (7). □

Theorem 2.11. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(g) < \infty$, $g(0) = 0$ and*

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)} = A, \text{ a real number } < \infty.$$

Then

$$\rho_h^{L^*}(f \circ g) \geqslant B \cdot \lambda_h^{L^*}(g).$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2 for all sufficiently large values of r that

$$\frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \left[re^{L(r)}\right]} \geqslant \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log \left[re^{L(r)}\right]},$$

$$i.e.,\ \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \left[re^{L(r)}\right]}$$
$$\geqslant \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log \mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}{\log \left[re^{L(r)}\right]},$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \left[re^{L(r)}\right]}$$
$$\geqslant \limsup_{r\to\infty} \left[\frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log \mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}{\log \left[re^{L(r)}\right]}\right],$$
$$i.e.,\ \limsup_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \left[re^{L(r)}\right]}$$
$$\geqslant \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)} \cdot \liminf_{r\to\infty} \frac{\log \mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}{\log \left[re^{L(r)}\right]},$$

$$i.e.,\ \rho_h^{L^*}(f \circ g) \geqslant B \cdot \lambda_h^{L^*}(g).$$

Thus the proof is complete. □

Theorem 2.12. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(g) \leqslant \rho_h^{L^*}(g) < \infty$, $g(0) = 0$ and*

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)} = B, \text{ a real number } < \infty.$$

Then

$$\lambda_h^{L^*}(f \circ g) \leqslant B \cdot \rho_h^{L^*}(g).$$

Theorem 2.13. *Let f, g and h be any three entire functions such that $0 < \rho_h^{L^*}(g) < \infty$, $g(0) = 0$ and*

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}(\mu_f(r))}{\log \mu_h^{-1}(r)} = A, \text{ a real number } < \infty$$

for a particular value of $\delta > 0$. Then

$$\rho_h^{L^*}(f \circ g) \geqslant A \cdot \rho_h^{L^*}(g).$$

The proof of Theorem 2.12 and Theorem 2.13 are omitted because those can be carried out in the line of Theorem 2.10 and Theorem 2.11 respectively.

Acknowledgements
The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

References

[1] L. Bernal, Orden relative de crecimiento de funciones enteras ,*Collect. Math.*, 39 (1988), 209-229.

[2] S. K. Datta and T. Biswas, Growth of entire functions based on relative order, *International Journal of Pure and Applied Mathematics*, 51 (1) (2009), 49-58.

[3] S. K. Datta and A. R. Maji, Relative order of entire functions in terms of their maximum terms, *Int. Journal of Math. Analysis*, 43 (5) (2011), 2119-2126.

[4] S. K. Datta, T. Biswas, and S. Ali, Growth estinmates of composite entire functions based on maximum terms using their relative L-order, *Advances in Applied Mathematical Analysis*, 7 (2) (2012), 119-134.

[5] S. K. Singh and G. P. Barker, Slowly changing functions and their applications, *Indian J. Math.*, 19 (1) (1977), 1-6.

[6] D. Somasundaram and R. Thamizharasi, A note on the entire functions of L-bounded index and L type, *Indian J. Pure Appl. Math.*, 19 (3) (1988), 284-293.

[7] A. P. Singh, On maximum term of composition of entire functions, *Proc. Nat. Acad. Sci. India*, 59 (A)(part I) (1989), 103-115.

[8] A. P. Singh and M. S. Baloria, On maximum modulus and maximum term of composition of entire functions, *Indian J. Pure Appl. Math.*, 22 (12) (1991), 1019-1026.

[9] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, Oxford, 1968.

[10] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company, 1949.

Sanjib Kumar Datta
Department of Mathematics
Assistant Professor of Mathematics
University of Kalyani
P.O.-Kalyani, Dist-Nadia, PIN-741235, West Bengal, India
E-mail: sanjib_kr_datta@yahoo.co.in

Tanmay Biswas
Research Scholar of Mathematics
Rajbari, Rabindrapalli
P.O.-Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India
E-mail: tanmaybiswas_math@rediffmail.com

Ahsanul Hoque
Department of Mathematics
Research Scholar of Mathematics
University Of Kalyani
P.O.-Kalyani, Dist-Nadia, PIN-741235, West Bengal, India
E-mail: ahoque033@gmail.com