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A General Norm on Extension of a Hilbert's Type Linear Operator

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Abstract. The main purpose of this paper is to study a general norm on extension of a Hilbert's type linear operator in the continuous and discrete form. In addition to expressing the norm of a Hilbert's type linear operator $T : L^2(0,\infty) \to L^2(0,\infty)$, a more general case with $\lambda > 0$, for the continuous form has been studied. By putting $\lambda = 1$ a norm of extension of Hilbert's integral linear operator is obtained. Similar results have been expressed for series when $0 < \lambda \leqslant 2$.

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1. Introduction and Preliminaries

If $f(t), g(t) \geqslant 0$, $0 < \int_0^\infty f^2(t)dt < \infty$, and $0 < \int_0^\infty g^2(t)dt < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \quad (1)$$

where the constant factor π is the best possible. Inequality (1) is named Hardy-Hilbert's integral inequality (see [1]). Under the same condition

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of (1), we have the Hardy-Hilbert's type inequality (see [1], Theorem 319, Theorem 341). similar to (1)that is

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < 4 \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \quad (2)$$

where the constant factor 4 is also the best possible. The corresponding inequalities for series are:

$$\sum_{n=1}^{\propto} \sum_{m=1}^{\propto} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\propto} a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\propto} b_n^2 \right\}^{\frac{1}{2}},$$

(3)

$$\sum_{n=1}^{\propto} \sum_{m=1}^{\propto} \frac{a_m b_n}{\max\{m,n\}} < 4 \left\{ \sum_{n=1}^{\propto} a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\propto} b_n^2 \right\}^{\frac{1}{2}},$$

where $\{a_n\}$ and $\{b_n\}$ are sequences such that $0 < \sum_{n=1}^\infty a_n^2 < \infty$, $0 < \sum_{n=1}^\infty b_n^2 < \infty$, and the constant factor π and 4 are both the best possible.

Let H be a real separable Hilbert space, and $T : H \to H$ be a bounded self-adjoint semi-positive definite operator, then (see [8]),

$$(a, Tb)^2 \leqslant \frac{\|T\|^2}{2} \left(\|a\|^2 \|b\|^2 + (a,b)^2 \right), \quad (4)$$

where $a, b \in H$ and $\|a\| = \sqrt{(a,a)}$ is the norm of a.
Set $H = L^2(0,\infty) = \left\{ f(x) : \int_0^\infty f^2(x)dx < \infty \right\}$ and define $T : L^2(0,\infty) \to L^2(0,\infty)$ as the following:

$$(Tf)(y) = \int_0^\infty \frac{1}{x+y} f(x)dx, \quad (5)$$

where $y \in (0,\infty)$. It is easy to see T is a bounded operator (see [7]). By (4), one has the sharper form of Hilbert's inequality as (see [8]),

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leqslant \frac{\pi}{\sqrt{2}} \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx + \left(\int_0^\infty f(x)g(x)dx \right)^2 \right\}^{\frac{1}{2}}, \quad (6)$$

In 2006 and 2007, Yang studied the Hilbert's inequalities by the norm of some Hilbert's type linear operators in the continuous and discrete forms (see [5,6]), and in the end of 2007 Li and his colleagues studied the Hilbert's type linear operators with the kernel $\frac{1}{A\min\{x,y\}+B\max\{x,y\}}$,(see [3]).
The main purpose of this article is to study the norm of Hilbert's type linear operator with the kernel $\frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^{\lambda}}$ such that $\lambda > 0$ for the continuous form, and $0 < \lambda \leqslant 2$ for the discrete form.

2. Main Results and Applications

Lemma 2.1. *Consider , $x, y > 0$, $A \geqslant 0$, $B > 0$, $\lambda > 0$, and $0 \leqslant \varepsilon < 1$, with the weight functions:*

$$\omega_\lambda(\varepsilon, x) := \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy, \quad (7)$$

$$\omega_\lambda(\varepsilon, y) := \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}\left(\frac{y}{x}\right)^{\frac{1+\varepsilon}{2}} dx,$$

where $\omega_\lambda(0,x) = \omega_\lambda(x)$, that is, $\omega_\lambda(\varepsilon,x) = \omega_\lambda(x) + o(1)$ $(\varepsilon \to 0^+)$. Then $0 < \omega_\lambda(x) = \omega_\lambda(y) < \infty$ is a constant.

Proof. For fixed x, letting $v = \frac{B}{A}(\frac{y}{x})$,$u = \frac{A}{B}(\frac{y}{x})$, we get

$$\omega_\lambda(\varepsilon, x) = \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy$$

$$= \int_0^x \frac{(xy)^{\frac{\lambda-1}{2}}}{(Ay+Bx)^\lambda}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy + \int_x^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{(Ax+By)^\lambda}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy \quad (8)$$

$$= \frac{B^{\frac{-(\varepsilon+\lambda)}{2}}}{A^{\frac{\lambda-\varepsilon}{2}}} \int_0^{\frac{A}{B}} \frac{u^{(\frac{-2+\lambda-\varepsilon}{2})}}{(1+u)^\lambda} du + \frac{A^{\frac{-(\varepsilon+\lambda)}{2}}}{B^{\frac{\lambda-\varepsilon}{2}}} \int_{\frac{B}{A}}^\infty \frac{v^{(\frac{-2+\lambda-\varepsilon}{2})}}{(1+v)^\lambda} dv.$$

$$\leqslant \frac{B^{\frac{-(\varepsilon+\lambda)}{2}}}{A^{\frac{\lambda-\varepsilon}{2}}}\int_0^\infty \frac{u^{(\frac{-2+\lambda-\varepsilon}{2})}}{(1+u)^\lambda}du + \frac{A^{\frac{-(\varepsilon+\lambda)}{2}}}{B^{\frac{\lambda-\varepsilon}{2}}}\int_0^\infty \frac{v^{(\frac{-2+\lambda-\varepsilon}{2})}}{(1+v)^\lambda}dv.$$

By Beta function (see [4]), one has

$$0 < \omega_\lambda(x) \leqslant \frac{2}{(AB)^{\frac{\lambda}{2}}}\beta(\frac{\lambda}{2},\frac{\lambda}{2}) < \infty.$$

Also in the same way:

$$\omega_\lambda(y) = \omega_\lambda(x) \leqslant \frac{2}{(AB)^{\frac{\lambda}{2}}}\beta(\frac{\lambda}{2},\frac{\lambda}{2}) < \infty.$$

Hence $0 < \omega_\lambda(x) = \omega_\lambda(y) < \infty$ is a constant. □

Lemma 2.2. *Consider, $m, n \in N$, $A \geqslant 0$, $B > 0$, $0 < \lambda \leqslant 2$, $0 \leqslant \varepsilon < 1$, and the weight function $w_\lambda(\varepsilon, n)$ for discrete forms as:*

$$w_\lambda(\varepsilon, n) := \sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}}}{[A\min\{m,n\} + B\max\{m,n\}]^\lambda}\left(\frac{n}{m}\right)^{\frac{1+\varepsilon}{2}}. \tag{9}$$

Then

$$w_\lambda(n) < \omega_\lambda(n). \tag{10}$$

Proof. It is obvious. □

Note 2.3. If $\lambda = 1$ then $\omega_\lambda(x) = D(A, B)$ given in ([3] lemma 1.2.).

3. A General Norm on Extension of a Hilbert's Type Linear Operator in the Continuous Forms

Theorem 3.1. *Consider, $A \geqslant 0$, $B > 0$, $\lambda > 0$, $T : L^2(0,\infty) \to L^2(0,\infty)$, and define:*

$$(Tf)(y) := \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f(x)dx \quad (y \in (0,\infty)), \tag{11}$$

Then, $\|T\| = \omega_\lambda(x)$ *is the general norm , and for any* $f(x), g(x) \geqslant 0$ *such that* $f, g \in L^2(0,\infty)$*, one has* $(Tf, g) < \omega_\lambda(x)\|f\|_2\|g\|_2$*, that is*

$$\int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f(x)g(y)dxdy < \omega_\lambda(x)\left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}, \tag{12}$$

where the constant factor $\omega_\lambda(x)$ *is the best possible.*

Proof. For $A \geqslant 0$, $B > 0$, applying Holder's inequality, we obtain

$$\begin{aligned}
(Tf,g) &= \left(\int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda} dx, g(y)\right) \\
&= \int_0^\infty \left[\int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda} dx\right] g(y)dy \\
&= \int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda} \left[f(x)\left(\frac{x}{y}\right)^{\frac{1}{4}}\right]\left[g(y)\left(\frac{y}{x}\right)^{\frac{1}{4}}\right] dxdy \\
&\leqslant \left\{\int_0^\infty \left[\int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}\left(\frac{x}{y}\right)^{\frac{1}{2}} dy\right] f^2(x)dx\right\}^{\frac{1}{2}} \\
&\times \left\{\int_0^\infty \left[\int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}\left(\frac{y}{x}\right)^{\frac{1}{2}} dx\right] g^2(y)dy\right\}^{\frac{1}{2}} \\
&= \left\{\int_0^\infty \omega_\lambda(x) f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \omega_\lambda(y) g^2(y)dy\right\}^{\frac{1}{2}} \\
&= \omega_\lambda(x)\left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(y)dy\right\}^{\frac{1}{2}} \\
&= \omega_\lambda(x)\|f\|_2\|g\|_2
\end{aligned} \tag{13}$$

and hence $\|T\| \leqslant \omega_{\lambda(x)}$. If (13) takes the form of the equality, then there exist constants α and β , not both zero such that (see [2])

$$\alpha f^2(x)\left(\frac{x}{y}\right)^{\frac{1}{2}} = \beta g^2(y)\left(\frac{y}{x}\right)^{\frac{1}{2}} \tag{14}$$

Therefore, we have

$$\alpha f^2(x)x = \beta g^2(y)y \qquad \text{a.e. on } (0,\infty)\times(0,\infty).$$

Hence there exists a constant c, such that

$$\alpha f^2(x)x = \beta g^2(y)y = c \qquad \text{a. e. on } (0,\infty)\times(0,\infty)$$

Without losing the generality, suppose $\alpha \neq 0$, then we obtain $f^2(x) = \frac{c}{\alpha x}$, a.e. on $(0,\infty)$, which contradicts the fact that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (13) takes the form of a strict inequality , and we obtain (3.2).

For any $a, b \geqslant 1$, $\varepsilon > 0$ sufficiently small, set $f_\varepsilon(x) = a^{\frac{\varepsilon}{2}} x^{\frac{-(1+\varepsilon)}{2}}$, if $x \in [a,\infty)$, $f_\varepsilon(x) = 0$, if $x \in (0,a)$. Similarly, $g_\varepsilon(y) = b^{\frac{\varepsilon}{2}} y^{\frac{-(1+\varepsilon)}{2}}$, if $y \in [b,\infty)$, and $g_\varepsilon(y) = 0$, if $y \in (0,b)$. Assume that the constant factor $\omega_\lambda(x)$ in (12) is not the best possible, then there exists a positive real number k with $k < \omega_\lambda(x)$ such that (12) is valid by changing $\omega_\lambda(x)$ to k. On one hand,

$$\int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f_\varepsilon(x) g_\varepsilon(y) dxdy < k\left\{\int_0^\infty f_\varepsilon^2(x)dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty g_\varepsilon^2(x)dx\right\}^{\frac{1}{2}} = \frac{k}{\varepsilon}. \qquad (15)$$

On the other hand, setting $t = \frac{y}{x}$, we have

$$\int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f_\varepsilon(x) g_\varepsilon(y) dxdy$$

$$= (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_{\frac{b}{x}}^\infty \frac{t^{\frac{\lambda-1}{2} - \frac{(1+\varepsilon)}{2}}}{[A\min\{1,t\} + B\max\{1,t\}]^\lambda} dtdx$$

$$= (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^\infty \frac{t^{\frac{\lambda-1}{2} - \frac{1+\varepsilon}{2}}}{[A\min\{1,t\} + B\max\{1,t\}]^\lambda} dtdx$$

$$-(ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{t^{\frac{\lambda-1}{2} - \frac{1+\varepsilon}{2}}}{[A\min\{1,t\} + B\max\{1,t\}]^\lambda} dtdx.$$

For $x \geqslant b$ and $0 < \varepsilon < 1$, we get

$$\int_0^{\frac{b}{x}} \frac{t^{(\frac{\lambda-1}{2} - \frac{1+\varepsilon}{2})}}{[A\min\{1,t\} + B\max\{1,t\}]^\lambda} dt = \int_0^{\frac{b}{x}} \frac{t^{(\frac{\lambda-1}{2} - \frac{1+\varepsilon}{2})}}{[At + B]^\lambda} dt, \qquad (16)$$

$$\leqslant \left(\frac{1}{B^\lambda}\right)\int_0^{\frac{b}{x}} t^{(\frac{\lambda-1}{2}-\frac{1+\varepsilon}{2})}dt \leqslant \left(\frac{1}{B^\lambda}\right)\left(\frac{2b^{\frac{\lambda}{2}}}{\lambda-1}\right)x^{\frac{1-\lambda}{2}}.$$

Thus

$$0<(ab)^{\frac{\varepsilon}{2}}\int_a^\infty x^{-(1+\varepsilon)}\int_0^{\frac{b}{x}}\frac{t^{\frac{\lambda-1}{2}-\frac{1+\varepsilon}{2}}}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda}dtdx$$

$$<\left(\frac{4}{B^\lambda}\right)\left(\frac{a^{\frac{2-\lambda}{2}}b^{\frac{1+\lambda}{2}}}{(\lambda-1)^2}\right)<\infty \tag{17}$$

Note that

$$(ab)^{\frac{\varepsilon}{2}}\int_a^\infty x^{-(1+\varepsilon)}\int_0^{\frac{b}{x}}\frac{t^{\frac{\lambda-1}{2}-\frac{1+\varepsilon}{2}}}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda}dtdx=O(1), \tag{18}$$

So we have

$$\int_0^\infty\int_0^\infty\frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}f_\varepsilon(x)g_\varepsilon(y)dxdy$$

$$=\frac{a^{\frac{-\varepsilon}{2}}b^{\frac{\varepsilon}{2}}}{\varepsilon}[\omega_\lambda(x)+o(1)]-O(1)$$

$$=\frac{a^{\frac{-\varepsilon}{2}}b^{\frac{\varepsilon}{2}}}{\varepsilon}[\omega_\lambda(x)+o(1)]. \tag{19}$$

Now from (15) and (19) we get $\frac{a^{\frac{-\varepsilon}{2}}b^{\frac{\varepsilon}{2}}}{\varepsilon}[\omega_\lambda(x)+o\ (1)]<\frac{k}{\varepsilon}$,that is, $\omega_\lambda(x)<k$ when ε is sufficiently small and $a,b\geqslant 1$, which contradicts the hypothesis. Hence the constant factor $\omega_\lambda(x)$ in (12) is the best possible and $\|T\|_2=\omega_\lambda(x)$. This completes the proof. □

Theorem 3.2. *If $A\geqslant 0$, $B>0$, $\lambda>0$, and $0<\int_0^\infty f^2(x)dx<\infty$. then*

$$\int_0^\infty\left[\int_0^\infty\frac{(xy)^{\frac{\lambda-1}{2}}f(x)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}dx\right]^2dy<$$

$$\omega_\lambda^2(x)\int_0^\infty f^2(x)dx, \tag{20}$$

where the constant factor $\omega_\lambda^2(x)$ is the best possible. Inequality (20) is equivalent to (12).

Proof. Let

$$g(y) = \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} dx.$$

Then , by (12), we get

$$0 < \int_0^\infty g^2(y)dy = \int_0^\infty \left[\int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} dx\right]^2 dy$$

$$= \int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f(x)g(y)dxdy$$

$$\leqslant \omega_\lambda(x) \left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(y)dy\right\}^{\frac{1}{2}}, \tag{21}$$

Hence, we obtain

$$0 < \int_0^\infty g^2(y)dy = \omega_\lambda^2(x) \left\{\int_0^\infty f^2(x)dx\right\} < \infty. \tag{22}$$

By (12), both (21) and (22) take the form of a strict inequality, so we have (20). On the other hand, suppose that (20) is valid. By Holder's inequality, we find

$$\int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)g(y)}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} dxdy$$

$$= \int_0^\infty \left[\int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} dx\right] g(y)dy$$

$$\leqslant \left\{\int_0^\infty \left[\int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} dx\right]^2 dy\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(y)dy\right\}^{\frac{1}{2}}.$$

By (20), we have (12). Thus (12) and (20) are equivalent. If the constant $\omega_\lambda^2(x)$ in (20) is not the best possible, then the constant $\omega_\lambda(x)$ in (12) is not the best possible. This completes the proof. $\square$

Note 3.3. If $A = B = 1$ and $\lambda = 1$ then by Theorem 3.2., one has

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y}dx\right]^2 dy < \pi^2 \int_0^\infty f^2(x)dx. \tag{23}$$

If $A = 0, B = 1$ and $\lambda = 1$, then one has

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx\right]^2 dy < 16 \int_0^\infty f^2(x)dx,$$

where the constant factors π^2 and 16 are both the best possible. Inequality (23) is Hilbert's inequality in continuous form.

4. The Corresponding Theorem for Series

Theorem 4.1. *Suppose that, $a_n, b_n \geqslant 0$, $A \geqslant 0$, $B > 0$, $0 < \lambda \leqslant 2$ and $0 < \sum_{n=1}^{\propto} a_n^2 < \infty$, $0 < \sum_{n=1}^{\propto} b_n^2 < \infty$, then*

$$\sum_{n=1}^{\propto}\sum_{m=1}^{\propto} \frac{(mn)^{\frac{\lambda-1}{2}}}{[A\min\{m,n\} + B\max\{m,n\}]^\lambda} a_m b_n < \omega_\lambda(n) \left(\sum_{n=1}^{\propto} a_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\propto} b_n^2\right)^{\frac{1}{2}}, \tag{24}$$

$$\sum_{n=1}^{\propto}\left[\sum_{m=1}^{\propto} \frac{(mn)^{\frac{\lambda-1}{2}} a_m}{[A\min\{m,n\} + B\max\{m,n\}]^\lambda}\right]^2 < \omega_\lambda^2(n) \sum_{n=1}^{\propto} a_n^2, \tag{25}$$

where the constant factors $\omega_\lambda(n)$ and $\omega_\lambda^2(n)$ are both the best possible and inequality (24) is equivalent to (25).

Proof. Using a method similar to Theorem 3.1., and applying Holder's inequality, we obtain

$$\sum_{n=1}^{\propto}\sum_{m=1}^{\propto} \frac{(mn)^{\frac{\lambda-1}{2}}}{[A\min\{m,n\} + B\max\{m,n\}]^\lambda} a_m b_n < \left\{\sum_{n=1}^{\propto} w_\lambda(n) a_n^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\propto} w_\lambda(n) b_n^2\right\}^{\frac{1}{2}}$$

By (25), we obtain (24).

For any $a, b \geqslant 1$, $\varepsilon > 0$ sufficiently small, setting $\tilde{a}_m = \left\{ a^{\frac{\varepsilon}{2}} m^{\frac{-(1+\varepsilon)}{2}} \right\}_{m=a}^{\infty}$, $\tilde{b}_n = \left\{ b^{\frac{\varepsilon}{2}} n^{\frac{-(1+\varepsilon)}{2}} \right\}_{n=b}^{\infty}$,then

$$\sum_{n=1}^{\propto} \sum_{m=1}^{\propto} \frac{(mn)^{\frac{\lambda-1}{2}}}{[A\min\{m,n\} + B\max\{m,n\}]^{\lambda}} \tilde{a}_m \tilde{b}_n$$
$$> \int_1^{\infty} \int_1^{\infty} \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\} + B\max\{x,y\}]^{\lambda}} f_\varepsilon(x) g_\varepsilon(y) dx dy, \qquad (26)$$

and

$$\left\{ \sum_{n=1}^{\propto} \tilde{a}_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\propto} \tilde{b}_n^2 \right\}^{\frac{1}{2}} = \sum_{n=a}^{\propto} \frac{a^\varepsilon}{n^{1+\varepsilon}}$$
$$< 1 + \int_a^{\infty} \frac{a^\varepsilon}{t^{1+\varepsilon}} dt = 1 + \frac{1}{\varepsilon}, \qquad (27)$$

If the constant factor $\omega_\lambda(n)$ in (24) is not the best possible, then applying the result of Theorem 3.1., we have a contradiction. Let

$$b_n = \sum_{m=1}^{\propto} \frac{(mn)^{\frac{\lambda-1}{2}}}{[A\min\{m,n\} + B\max\{m,n\}]^{\lambda}} a_m.$$

We can obtain the following relation:

$$\sum_{n=1}^{\propto} \left[\sum_{m=1}^{\propto} \frac{(mn)^{\frac{\lambda-1}{2}} a_m}{[A\min\{m,n\} + B\max\{m,n\}]^{\lambda}} \right]^2 = \sum_{n=1}^{\propto} b_n^2$$
$$= \sum_{n=1}^{\propto} \sum_{m=1}^{\propto} \frac{(mn)^{\frac{\lambda-1}{2}}}{[A\min\{m,n\} + B\max\{m,n\}]^{\lambda}} a_m b_n$$

Applying (24) and a method similar to Theorem 3.2., we conclude that (25), and (25) are equivalent to (24) with the best constant. $\square$

Note 4.2. If $A = B = 1$ and $\lambda = 1$ then by Theorem 4.1., one has Hilbert's inequality in discrate form, as:

$$\sum_{n=1}^{\propto} \sum_{m=1}^{\propto} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\propto} a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\propto} b_n^2 \right\}^{\frac{1}{2}}.$$

If $A = 0, B = 1$ and $\lambda = 1$, then one has

$$\sum_{n=1}^{\propto} \sum_{m=1}^{\propto} \frac{a_m b_n}{\max\{m, n\}} < 4 \left\{ \sum_{n=1}^{\propto} a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\propto} b_n^2 \right\}^{\frac{1}{2}},$$

where the constant factors π and 4 are both the best possible.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

[2] J. Kuang, *Applied inequalities*, Shandong science press, Jinan, 2003.

[3] Y. Li, Z. Wang, and B. He, Hilbert's type linear operator and some extensions of Hilbert's inequality, *J. Math. Anal. Appl.,* 2007 (2007), 1-10.

[4] Z. Wang and D. Gua, *An introduction to special functions,* Science press, Beijing, 1979.

[5] B. Yang, On the norm of a Hilbert's type linear operator and applications, *J. Math. Anal. Appl.*, 325 (1) (2007), 529-541.

[6] B. Yang, On the norm of an integral operator and applications, *J. Math. Anal. Appl.*, 321 (1) (2006), 182-192.

[7] B. Yang and T. M. Rassias, On a new extension of Hilbert's inequality, *J. Math. Inequal. Appl.*, 8 (4) (2005), 575-582.

[8] K. Zhang and A. bilinear inequality, *J. Math. Anal. Appl.*, 271 (1) (2002), 288-296.

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