An Application of Linear Algebra over Lattices

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Abstract. In this paper, first we consider $L^n$ as a semimodule over a complete bounded distributive lattice $L$. Then we define the basic concepts of module theory for $L^n$. After that, we proved many similar theorems in linear algebra for the space $L^n$. An application of linear algebra over lattices for solving linear systems, was given.

AMS Subject Classification: 06DXX; 15A03; 16D10; 06F05.

Keywords and Phrases: Lattices, semimodule, linear algebra, linear system.

1. Introduction

Fuzzy linear systems of equations and inequalities over a bounded chain have been studied by many authors [6], [8], [7]. To extend this concept to $L$-fuzzy linear systems over a bounded distributive lattice $L$, we need some basic definitions of linear algebra over lattices such as linearly independent subset, a subsemimodule generated by a set and so on. For more details see [3], [2]. By defining subsemimodule generated by a set, we can find a theoretical necessary and sufficient condition for
consistency of the linear system of equations $A \ast X = b$ over a bounded distributive lattice.

**Definition 1.1.** Let $(H, \ast)$ be a commutative semigroup (or monoid) with a reflexive and transitive order $\leq$ on it. $(H, \ast, \leq)$ is called an ordered commutative semigroup (or monoid) if

$$a \leq b \implies a \ast c \leq b \ast c \quad \forall a, b, c \in H.$$ 

**Definition 1.2.** Let $(H, \ast)$ be a commutative group (resp. semigroup, monoid) with a partial order $\leq$. $(H, \ast, \leq)$ is called a lattice-ordered commutative group (resp. semigroup, monoid), if

$$a \leq b \implies a \ast c \leq b \ast c, \quad \forall a, b, c \in H.$$ 

For simplicity, we call it l-group (resp. l-semigroup, l-monoid).

**Example 1.3.** Every lattice $(L, \leq)$ is a l-semigroup, by letting $\ast = \wedge$. Clearly a bounded lattice is a l-monoid in this way.

**Definition 1.4.** Let $\text{Mat}_{n \times m}(L)$ be the set of all $n \times m$ matrices over the lattice $(L, \leq)$. Define a partial order relation on $\text{Mat}_{n \times m}(L)$ as follows:

$$X \leq Y \iff x_{ij} \leq y_{ij}; \quad \text{for all } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m,$$

where $X, Y \in \text{Mat}_{n \times m}(L)$. One can see that $(\text{Mat}_{n \times m}(L), \leq)$ is a lattice where its supremum and infimum are defined componentwise on $\text{Mat}_{n \times m}(L)$ induced by the supremum and infimum of lattice $L$, respectively.
Definition 1.5 ([10]). Let \((R, \oplus)\) be a commutative monoid with neutral element 0 and \((R, \otimes)\) be a monoid with neutral element 1 where 0 \(\neq\) 1. Then, \((R, \oplus, \otimes)\) is called a semiring with unity 1 and zero 0, if for all \(a, b, c \in R\), the following conditions hold:

(a) \(a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)\),

(b) \((b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)\),

(c) \(0 = a \otimes 0 = 0 \otimes a\).

Example 1.6. Let \(L\) be a bounded distributive lattice. Then, \((L, \lor, \land)\) and \((L, \land, \lor)\) are semirings.

Definition 1.7 ([10]). \((R, \oplus, \otimes, \leq)\) is called an ordered semiring if

(a) \((R, \oplus, \otimes)\) is a semiring,

(b) \((R, \oplus, \leq)\) is an ordered commutative monoid,

(c) for all \(a, b, c, d \in R\),

(i) \(a \leq b\) and \(c \geq 0 \implies a \otimes c \leq b \otimes c\) and \(c \otimes a \leq c \otimes b\),

(ii) \(a \leq b\) and \(d \leq 0 \implies a \otimes d \geq b \otimes d\) and \(d \otimes a \geq d \otimes b\).

Definition 1.8 ([10]). Let \((H, \ast, \leq)\) be a commutative ordered monoid with neutral element \(e\) and let \((R, \oplus, \otimes)\) be a semiring with unity 1 and zero 0.

Moreover, suppose that \(: R \times H \rightarrow H\) is a scalar multiplication such that for all \(\alpha, \beta \in R\) and for all \(a, b \in H\):

(a) \((\alpha \otimes \beta) \cdot a = \alpha \cdot (\beta \cdot a)\),
(b) \((\alpha \oplus \beta).a = (\alpha.a) \oplus (\beta.a)\),
(c) \(\alpha.(a \ast b) = (\alpha.a) \ast (\alpha.b)\),
(d) \(0.a = e\),
(e) \(1.a = a\),
then, \((R, \oplus, \otimes, H, \ast, .)\) is called an ordered semimodule over \(R\).

**Remark 1.9.** Let \(L\) be a bounded distributive lattice.

Then, \((L, \lor, \land, L, \lor, \land)\) and \((L, \land, \lor, L, \land, \lor)\) are semimodules over \((L, \lor, \land)\) and \((L, \land, \lor)\), respectively.

Upward and downward sets, as important notions in optimization (see [4], [5]), are used in [9] as in the following definition.

**Definition 1.10.** Let \((L, \leq)\) be a lattice.

(i) A subset \(U \subseteq L\) is called upward set if \((a \in U, x \geq a) \implies x \in U\).

(ii) A subset \(D \subseteq L\) is called downward set if \((a \in D, x \leq a) \implies x \in D\).

**Example 1.11.** Let \((L, \leq)\) be a lattice and \(a \in L\). Then \(\{x \in L | x \geq a\}\) is an upward set and \(\{x \in L | x \leq a\}\) is a downward set.

We can easily prove the following proposition.

**Proposition 1.12.** Let \((L, \leq)\) be a lattice and \(M_i \subseteq L\) for \(i \in I\). Then \(\bigcup_{i \in I} M_i\) is an upward (resp. downward) set if each \(M_i\); \(i \in I\) is upward (resp. downward) set.
2. Basis For Semimodules

In this section we need to extend some basic definition of linear algebra to concepts of lattices. In this case suppose $L$ is a complete distributive lattice and consider $L^n$ as $Mat_{n\times1}(L)$, the set of all $n \times 1$ matrices over $L$. By Definition 1.4., $L^n$ is a lattice. Clearly $L^n$ is a distributive complete lattice if $L$ is so. For every bounded distributive lattice $L$, $(L, \vee, \wedge)$ is a semiring by Example 1.6. and hence $(L^n, \wedge, \leq)$ is a lattice-ordered commutative monoid, by Example 1.3. So we can construct a semimodule as follows.

**Theorem 2.1.** Let $L$ be a distributive complete lattice. Then $(L^n, \vee, \leq)$ is a semimodule over $(L, \vee, \wedge)$.

**Proof.** Let $L$ be a bounded distributive lattice. Then $(L^n, \vee, \leq)$ is a semimodule over $(L, \vee, \wedge)$ with scalar multiplication $\bar{\wedge}$ defined by $\bar{\wedge} : L \times L^n \rightarrow L^n$ such that

$$\alpha \bar{\wedge} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \alpha \wedge a_1 \\ \alpha \wedge a_2 \\ \vdots \\ \alpha \wedge a_n \end{pmatrix},$$

which for simplification, we write it as $\wedge$.

In this way $(L^n, \vee, \leq)$ satisfies all conditions of Definition 1.8. Note that
the identity element of \((L^n, \lor)\) is a column matrix which all of its entry are equal to 0. □

**Definition 2.2.** Let \((H, *, \leq)\) be a semimodule over semiring \((R, \oplus, \otimes)\) and \(K\) be a subset of \(H\) such that \((K, *, \leq)\) is a monoid. Then \((K, *, \leq)\) is called a subsemimodule of \((H, *, \leq)\) if it is a semimodule over \((R, \oplus, \otimes)\) and it is denoted by \(K \leq_m H\).

The following theorem can be proved easily.

**Theorem 2.3.** Let \((H, *, \leq)\) be a semimodule over semiring \((R, \oplus, \otimes)\) and \(K\) be a subset of \(H\). Then \(K \leq_m H\) if and only if

(i) \(e \in K\)

(ii) \(x * y \in K\) for all \(x, y \in K\),

(iii) \(a.x \in K\) for all \(a \in R\), and \(x \in K\).

**Corollary 2.4.** Let \(L\) be a distributive complete lattice and \(K\) be a sublattice of \(L\) which contains 0. Then \((K^n, \lor, \leq)\) is a semimodule over \((L, \lor, \land)\) if and only if for every elements \(x \in L\) and \(y \in K\), we have \(x \land y \in K\).

**Example 2.5.** Let \(L = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}\) and \(x \leq y\) if \(x\) divides \(y\). Consider the sublattice \(K = \{1, 2, 3, 6\}\). Then, \(L\) and \(K\) satisfy on Corollary 2.4. Hence \((K^n, \lor, \leq)\) is a semimodule over \((L, \lor, \land)\).

**Definition 2.6.** Let \((H, *, \leq)\) be a semimodule over \((R, \oplus, \otimes)\) and \(X\) be
a subset of $H$.

(i) The subsemimodule hull of (or subsemimodule generated by) $X$ is the intersection of all subsemimodules of $H$ which contains $X$ and denoted by $<X>$. Hence

$$<X> = \bigcap_{X \subseteq K \subseteq H} K.$$ 

In the other words, $<X>$ is the smallest subsemimodule of $H$ which contains $X$.

(ii) The upward hull of (or upward set generated by) $X$ is defined as the intersection of all upward subsets of $H$ which contains $X$ and is denoted by $<X^*>$. So, $<X^*> = \bigcap \{K : X \subseteq K \text{ and } K \text{ is an upward subset of } H\}$. In the other words, $<X^*>$ is the smallest upward subset of $H$ which contains $X$.

(iii) The downward hull of (or downward set generated by) $X$ is defined as the intersection of all downward subsets of $H$ which contains $X$ and is denoted by $<X_*>$. So, $<X_*> = \bigcap \{K : X \subseteq K \text{ and } K \text{ is a downward subset of } H\}$. In the other words, $<X_*>$ is the smallest downward subset of $H$ which contains $X$.

**Lemma 2.7.** Let $(H, *, \leq)$ be a semimodule over $(R, \oplus, \otimes)$ and $x \in H$. Then,

(i) $<\{x\}^*> = \{a \in H : a \geq x\}$, and

(ii) $<\{x\}_*> = \{a \in H : a \leq x\}$. 
Definition 2.8. Let \((H, *, \leq)\) be a semimodule over semiring \((R, \oplus, \otimes)\) with scalar multiplication "." and \(X\) be a subset of \(H\). By a linear combination of elements \(x_1, \ldots, x_m \in X\), we mean \((a_1.x_1)^* \cdots (a_m.x_m)^*\) where \(a_1, \ldots, a_m \in R\) and \(m\) is a positive integer.

Theorem 2.9. Let \((H, *, \leq)\) be a semimodule over \((R, \oplus, \otimes)\) and \(X\) be a subset of \(H\).

(i) Consider \(M = \{(a_1.x_1)^* \cdots (a_m.x_m)^* | x_1, \ldots, x_m \in X, a_1, \ldots, a_m \in R\}\); as the set of all finite linear combinations of elements of \(X\). Then, \(<X> = M\).

(ii) \(<X^*> = \bigcup_{x \in X} <\{x\}^*>\).

(iii) \(<X_*> = \bigcup_{x \in X} <\{x\}^*>\).

Proof. The proofs of (i)-(iii) follow from Lemma 2.7. Definition 2.8. and Proposition 1.12. \(\Box\)

Example 2.10. Let \(L = [0, 10]\); the bounded chain of real numbers between 0 and 10. Consider semimodule \((L^2, \lor, \land)\) over \((L, \lor, \land)\), where \(\leq\) is usual partial order on \(L\). For \(X_1 = \{(2, 3)^T, (5, 1)^T\}\) the subsemimodule generated by \(X_1\) is shown in Fig. 1.
Fig. 1. Subsemimodule hull of $X_1$

The upward hull of $X_1$ is shown in Fig. 2.

Fig. 2. Upward hull of $X_1$

The downward hull of $X_1$ is shown in Fig. 3.
Now consider $X_2 = \{(2, 4)^T, (5, 9)^T\}$. The subsemimodule hull of $X_2$ is shown in Fig. 4.

The subsemimodule $< X_3 >$, where $X_3 = \{(3, 1)^T, (5, 2)^T, (2, 4)^T\}$,
is as follows:

![Diagram of subsemimodule hull of X₃]

Fig. 5. Subsemimodule hull of $X₃$

**Definition 2.11.** Let $(H, *, \leq)$ be a semimodule over $(R, \oplus, \otimes)$ with zero 0. A subset $X$ of $H$ is called linearly independent if for all finite subset \( \{x_1, \ldots, x_m\} \subseteq X \), and elements $a_1, \ldots, a_m \in R$; \( (a_1.x_1) \ast \ldots \ast (a_m.x_m) = e \) imply $a_1 = \ldots = a_m = 0$.

If the subset $X$ is not linearly independent, it is called linearly dependent.

**Example 2.12.** Let $L = \{1, 2, 3, 6\}$ and $x \leq y$ means that $x$ divides $y$. Clearly $(L, \lor, \leq)$ is a semimodule over $(L, \lor, \land)$ with zero 1. Since $2 \land 3 = 1$, the set \{3\} is not linearly independent.

**Remark 2.13.** By the previous example, it is not true that if $x \neq 0$ then \{x\} is linearly independent. But if $L$ is a chain, then for every
non-zero element \( x \), the set \( \{ x \} \) is linearly independent.

**Definition 2.14.** Let \( (H, *, \leq) \) be a semimodule over \( (R, \oplus, \otimes) \). A linearly independent subset \( B \) of \( H \) is called a basis for \( H \) over \( R \), if \( \langle B \rangle = H \).

**Example 2.15.** Let \( L \) be as in Example 2.5. (see Fig. 6).

In this lattice the following subsets of \( L \) are linearly independent:

- \( K_1 = \{6\} \), \( K_2 = \{6, 12\} \), \( K_3 = \{12, 18\} \)
- \( K_4 = \{6, 12, 36\} \), \( K_5 = \{6, 12, 18, 36\} \)

But the following subsets are linearly dependent:

- \( K_6 = \{9\} \), \( K_7 = \{2, 3\} \), \( K_8 = \{4, 9\} \), \( K_9 = \{6, 9\} \)

Some sublattices generated by above subsets of \( L \) are as follows:

- \( \langle K_9 \rangle = \{1, 2, 3, 6, 9, 18\} \), \( \langle K_3 \rangle = \langle K_4 \rangle = \langle K_5 \rangle = \langle K_8 \rangle = L \)
- \( \langle K_6 \rangle = \{1, 3, 9\} \)

Clearly \( K_3, K_4 \) and \( K_5 \) are bases of \( L \). Also

- \( \langle (K_9)_* \rangle = \{1, 2, 3, 6, 9\} \), \( \langle K_9^* \rangle = \{6, 9, 12, 18, 36\} \)
- \( \langle (K_5)_* \rangle = L \), \( \langle K_5^* \rangle = K_5 \).
Remark 2.16. (i) Note that although \( < K_8 > = L \), but \( K_8 \) contains no linearly independent subset.

(ii) For the basis \( K_3 \) we have \( 6 = (6 \land 12) \lor (6 \land 18) = (2 \land 12) \lor (3 \land 18) = (3 \land 12) \lor (2 \land 18) \). Therefore, representation of any elements of \( L \) in terms of a linear combination of elements of a basis is not unique.

Example 2.17. Suppose \((L, \leq)\) be a bounded distributive lattice. Clearly, \( \{1\} \) is a basis for \((L, \land, \leq)\) over \((L, \land, \lor)\). Note that in semimodule \((L^2, \land, \leq)\), the set \( \{(1, 1)^T\} \) is linearly independent but \( < \{(1, 1)^T\} > \neq L^2 \).

3. Consistency of \( A \ast X = b \).

In this section we consider semimodule \((H, \ast, \leq)\) over semiring \((R, \oplus, \otimes)\).

By a linear system of equations \( A \ast X = b \) over \( R \) we mean the following equations:

\[
\begin{align*}
(a_{11} \ast x_1) \ast (a_{12} \ast x_2) \ast \ldots \ast (a_{1n} \ast x_n) &= b_1 \\
(a_{21} \ast x_1) \ast (a_{22} \ast x_2) \ast \ldots \ast (a_{2n} \ast x_n) &= b_2 \\
& \quad \vdots \\
(a_{m1} \ast x_1) \ast (a_{m2} \ast x_2) \ast \ldots \ast (a_{mn} \ast x_n) &= b_m
\end{align*}
\]

where \( a_{ij} \in R \) and \( x_i, b_j \in H \) for all \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \).
Theorem 3.1. Let $L$ be a bounded distributive lattice. Consider $(L^n, \vee, \leq)$ as a semimodule over semiring $(L, \vee, \wedge)$ with scalar multiplication "$\wedge$". Let $A$, $X$ and $b$ are $m \times n$, $n \times 1$ and $m \times 1$ matrices over $L$, respectively. The linear system $A \vee X = b$ has a solution if and only if $b$ belongs to the subsemimodule generated by columns of $A$.

Proof. If we show the columns of $A$ by $A_1, A_2, ..., A_n$; then the linear system $A \vee X = b$ can shown by

$$(x_1 \wedge A_1) \vee (x_2 \wedge A_2) \vee ... \vee (x_n \wedge A_n) = b$$

and clearly the linear system has a solution if and only if $b \in <\{A_1, ..., A_n\}>$ by Theorem 2.9. □

Example 3.2. Let $L$, $K_9$ and $K_8$ be as in Example 2.15. consider the linear equation

$$(6 \wedge x_1) \vee (9 \wedge x_2) = 3$$

Then the set of all solutions of (1) is

$$\{(1,3)^T, (1,6)^T, (1,12)^T, (3,1)^T, (3,3)^T, (3,6)^T, (3,12)^T, (3,2)^T, (3,4)^T, (9,1)^T, (9,3)^T, (9,6)^T, (9,12)^T, (9,2)^T, (9,4)^T\}.$$ 

Linear equation (1) has solution since $3 \in <K_9>$; the subsemimodule generated by $\{6, 9\}$. But if we change right hand side of (1) to 12 we have:

$$(6 \wedge x_1) \vee (9 \wedge x_2) = 12$$

(2)
Clearly (2) doesn’t have any solution since $12 \not\in K_9$. Now consider

$$(4 \land x_1) \lor (9 \land x_2) = b$$

(3)

Since $<\{4,9\}> = <K_8> = L$, so (3) has solution for all $b \in L$.

**Remark 3.3.** Note that Theorem 3.1. gives a theoretical necessary and sufficient condition for consistency of (*)&. A computational necessary and sufficient condition for consistency of (*) over a bounded chain was given in [6]. Finding such a condition(s) over a bounded distributive lattice is still an open problem.

**References**


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