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An Error Bound for Solution of Fredholm Integral Equations by Adomian Method

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Abstract. In this paper, we will obtain an efficient computable upper bound for approximate solution of linear Fredholm integral equations obtained by Adomian decomposition method. Numerical examples are presented to show the effectiveness of the upper bounds.

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1. Introduction

The Adomian decomposition method was developed by G. Adomian in the beginning of the 1980's [3, 4]. This method has been used for solving algebraic, differential, integro-differential, differential-delay, and partial differential equations. The solution is usually represented as an infinite series which converges to accurate solution. Despite of classical methods, Adomian decomposition method does not need to have some techniques and assumptions which may change the underlying problem seriously such as linearization or perturbation. The reader is referred to [2, 11, 14] for undefined notations and terminology and more details. In many papers [1, 9, 10] convergence of Adomian method have been discussed by various techniques. To review the standard Adomian decomposition method, refer to [8, 12, 13].
In this paper we present a satisfactory posterior error for approximate

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solution produced by Adomian method.
This paper is organized as follows. In section 2, a brief description of Adomian method is presented. A posterior error for approximate solution of integral equations of the second kind obtained by Adomian method is discussed in Section 3. In section 4, some examples are illustrated to applied the posterior error and compare the solutions. In conclusion we give an interpretation of the results.

2. Outline of Adomian Decomposition Method

We consider the functional equation

$$u = f + Lu + Nu, \tag{1}$$

where L is a linear operator, N represents an analytical nonlinear operator, and f is the source term. For nonlinear equations, the nonlinear term Nu is usually represented by an infinite series of the so-called Adomian polynomials

$$N(u) = \sum_{k=o}^{\infty} A_k, \tag{2}$$

where A_k 's are generated for all kinds of nonlinearity and obtained by

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} [N(\sum_{i=0}^{\infty} u_i \lambda^i)]_{|\lambda=0}, \quad k = 0, 1, 2, \cdots . \tag{3}$$

Specific algorithms were set in [5, 6, 13] to formulate Adomian polynomials. In the following, we sketch the basic principles of the standard Adomian decomposition method. The standard Adomian method defines the solution u by the series

$$u = \sum_{n=0}^{\infty} u_n, \tag{4}$$

where the components $u_0, u_1, u_2, \ldots$ are usually determined recursively. If we substitute (2) and (4) into (1), we will have

$$\sum_{n=0}^{\infty} u_n = f + L \sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n. \tag{5}$$

Leading to:

$$u_0 = f,$$

$$u_{k+1} = L(u_k) + A_k(u_0, u_1, \ldots, u_k), \quad k \geqslant 0. \tag{6}$$

Substituting A_k into (5) leads to the determination of the components of u. Having determined the components $u_0, u_1, u_2, \ldots$ the solution u in a series form defined by (4) follows immediately.

3. An Error Bound for Approximate Solution Using ADM

We consider numerical solution of linear Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \quad a \leqslant x \leqslant b \tag{7}$$

which we can write it in the form

$$u = f + \lambda k u, \tag{8}$$

where $k(x,t)$ and $f(x)$ are known L^2 functions, and $u(x)$ is to be determined.
The Adomian technique consists of representing $u(x)$ as a series

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{9}$$

Now if we substitute (9) in the relation (7), we will have

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b k(x,t) \sum_{n=0}^{\infty} u_n(t)dt.$$

Note that the Adomian method uses the following recursive relations

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = \lambda \int_a^b k(x,t)u_n(t)dt, \quad n = 0, 1, 2, \cdots \tag{10}$$

which we can write it in the form

$$u_0 = f,$$

$$u_{n+1} = \lambda k u_n, \quad n = 0, 1, 2, \cdots . \tag{11}$$

The following theorem presents foregoing upper bound.

Theorem 3.1. *Let*

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

be the solution obtained by Adomian method for equation (7) and

$$\overline{u}_m = \sum_{n=0}^{m} u_n, \tag{12}$$

be an approximation of u*. If* $\|\lambda k\| < 1$ *and*

$$e_m = \overline{u}_m - u \tag{13}$$

then

$$\|e_m\| \leqslant \frac{\|u_{m+1}\|}{1 - \|\lambda k\|}. \tag{14}$$

Proof. Since $e_m = \overline{u}_m - u$ then

$$e_m = e_{m+1} - u_{m+1}. \tag{15}$$

Also from equation(11) we have

$$u_1 = \lambda k u_0$$

$$u_2 = \lambda k u_1$$

$$\vdots$$

$$u_{m+1} = \lambda k u_m.$$

Summing up the above equations, we have

$$\sum_{n=1}^{m+1} u_n = \sum_{n=0}^{m} \lambda k u_n = \lambda k \sum_{n=0}^{m} u_n.$$

Now using (12)

$$\overline{u}_{m+1} - u_0 = \lambda k \overline{u}_m. \tag{16}$$

Since u is the solution of (8) thus

$$u = f + \lambda k u. \tag{17}$$

By subtracting (17) from (16) we have

$$\overline{u}_{m+1} - u - u_0 = \lambda k \overline{u}_m - f - \lambda k u.$$

Using $u_0 = f$, we conclude

$$\overline{u}_{m+1} - u = \lambda k(\overline{u}_m - u)$$

and, by (13)

$$e_{m+1} = \lambda k e_m. \tag{18}$$

Next, from (15) we have

$$\|e_m\| = \|e_{m+1} - u_{m+1}\| \leqslant \|e_{m+1}\| + \|u_{m+1}\|.$$

Now using (18), we have

$$\|e_m\| \leqslant \|\lambda k e_m\| + \|u_{m+1}\|$$

thus

$$\|e_m\| \leqslant \frac{\|u_{m+1}\|}{1 - \|\lambda k\|},$$

and the proof is complete. □

Corollary 3.2. *Suppose all the assumptions in Theorem (3.1.) hold, then*

$$\|e_m\| \leqslant \frac{\|\overline{u}_{m+1} - \overline{u}_m\|}{1 - \|\lambda k\|}. \tag{19}$$

Proof. By hypothesis $u_{m+1} = \overline{u}_{m+1} - \overline{u}_m$ and the proof is trivial. □

4. Numerical Examples

Now we illustrate the above conclusion by some examples.

Example 1. Consider the following integral equation:

$$u(x) = \frac{23}{24}\cos x + \int_0^{\frac{\pi}{6}} (\sin^2 t \cos x) u(t)dt, \quad 0 \leqslant x \leqslant \frac{\pi}{6},$$

with the exact solution $u(x) = \cos x$.
Using Adomian method, we have

$$u_n(x) = \frac{23}{24^{n+1}}\cos x, \quad n = 0, 1, 2, \cdots .$$

Thus $\overline{u}_5 = \sum_{n=0}^{5} u_n = \frac{23}{24}(1 + \frac{1}{24} + \cdots + \frac{1}{24^5})\cos x$

$= (1 - \frac{1}{24^6})\cos x = 0.999999994767 \cos x$

and

$$e_5 = 0.5232780886 \times 10^{-8} \cos x.$$

Therefore

$$\|e_5\|_E \leqslant 0.5232780886 \times 10^{-8}.$$

On the other hand, by the theorem $\|e_5\|_E \leqslant \frac{\|u_6\|_E}{1-\|\lambda k\|_E}$.
Since $\|\lambda k\|_E = 0.057475$ we conclude

$$\|e_5\|_E \leqslant \frac{\|\frac{23}{24^7}\cos x\|_E}{1 - 0.057475} \leqslant \frac{\frac{23}{24^7}}{1 - 0.057475} = 0.532055 \times 10^{-8}.$$

We observe that the posterior error is quite satisfactory.

Example 2. Consider the following integral equation:

$$u(x) = \cos x + \int_0^{\frac{\pi}{6}} u(t)\cos x^2 dt.$$

Apply the decomposition method, then we have the following

$$u_0(x) = \cos x,$$

$$u_1(x) = \frac{1}{2}\cos x^2,$$

$$u_2(x) = \int_0^{\frac{\pi}{6}} \cos x^2 \frac{\cos t^2}{2} dt \simeq 0.259839 \cos x^2,$$

$u_3(x) \simeq 0.135032 \cos x^2,$
$u_4(x) \simeq 0.070173 \cos x^2,$
$u_5(x) \simeq 0.0364673 \cos x^2,$
$u_6(x) \simeq 0.0189512 \cos x^2,$
$u_7(x) \simeq 0.0098485 \cos x^2,$
$u_8(x) \simeq 0.00511804 \cos x^2,$
$u_9(x) \simeq 0.00265973 \cos x^2,$
$u_{10}(x) \simeq 0.0013822 \cos x^2,$
$u_{11}(x) \simeq 0.000718297 \cos x^2,$
$u_{12}(x) \simeq 0.000373282 \cos x^2,$
$u_{13}(x) \simeq 0.000193986 \cos x^2,$
$u_{14}(x) \simeq 0.0001008 \cos x^2,$
$u_{15}(x) \simeq 0.0000523886 \cos x^2,$
$\overline{u}_{14}(x) \simeq \cos x + 1.040857335 \cos x^2,$
$\|k\|_E \simeq 0.2700914619,$

$$\|e_{14}\| \leqslant \frac{|0.000052 \cos x^2|}{1 - 0.27009146} \leqslant \frac{0.000052}{0.729908} = 0.000071$$

Remark. *In Example 2., the exact solution is not known and for $i \geqslant 2$, u_i is obtained by quadrature rule [7].*

Example 3.

$$u(x) = -7x + \int_0^2 3xtu(t)dt \quad 0 < x < 2,$$

with the exact solution $u(x) = x$.
In this example $\|k\|_E = 8$ and posterior error is not applicable. On the other hand, if we use Adomian method, we have
$u_0(x) = -7x$
$u_1(x) = -56x = -7 \times 8x$

$u_2(x) = -448x = -7 \times 8^2 x$
$u_n(x) = -7 \times 8^n x,$
It shows that Adomian method is not convergent.

5. Conclusion

Inequality (14) represents a computable error bound, provided that $\|\lambda k\|$ is known or can be estimated and also $\|\lambda k\| < 1$. Computationally, often we can estimate $\|\lambda k\|$ rather easily, also $\|u_{m+1}\|$ is computable in the course of Adomian process. Inequality (14) is useless if $\|\lambda k\| \geqslant 1$ although $\|\lambda k\| < 1$ is not a necessary condition for convergence of the series $u = \sum_{n=0}^{\infty} u_n$.

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