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Some Results on Graded Generalized Local Cohomology Modules

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Abstract. Let $R = \oplus_{n \geqslant 0} R_n$ be a graded Noetherian ring with local base ring R_0 and let $R_+ = \oplus_{n \geqslant 1} R_n$. Let M and N be finitely generated graded R-modules. In this paper we extend some of the known results about ordinary local cohomology modules $H^i_{R_+}(M)$ to generalized local cohomology modules $H^i_{R_+}(M, N)$. Indeed, among other things, we prove that certain submodules and factor modules of $H^i_{R_+}(M, N)$ are Artinian for some i. Also we obtain some results on the asymptotic behaviour of the n-th graded components $H^i_{R_+}(M, N)_n$ of $H^i_{R_+}(M, N)$ for $n \longrightarrow -\infty$.

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1. Introduction

There is a lot of current interest in the theory of graded local cohomology modules and, in recent years, there have appeared many papers in this area of research. The main purpose of this paper is to extend some of the known results about ordinary graded local cohomology to the generalized local cohomology.

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For an ideal $\mathfrak{a}$ of a commutative Noetherian ring R and R-modules M and N, the i-th generalized local cohomology module

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\geqslant 1} Ext^i_R(\frac{M}{\mathfrak{a}^n M}, N),$$

was introduced by Herzog in [8] and studied further in [1,2,17,18]. It is clear that if $M = R$, then $H^i_{\mathfrak{a}}(M,N)$ is converted to $H^i_{\mathfrak{a}}(N)$, the i-th ordinary local cohomology module of N with respect to $\mathfrak{a}$.
Throughout the paper we assume that $R = \bigoplus_{n\geqslant 0} R_n$ is a positive graded commutative Noetherian ring and that $R_+ = \bigoplus_{n>0} R_n$ is the irrelevant graded ideal of R. Also we use $M = \oplus_{n\in\mathbb{Z}} M_n$ and $N = \oplus_{n\in\mathbb{Z}} N_n$ to denote non-zero finitely generated graded R-modules. (Here $\mathbb{Z}$ denotes the set of all integers.)
It is well known, see for example [9], that $H^i_{R_+}(M,N)$ carry a natural grading and that the grading of it have some similar properties as the ordinary graded local cohomology module $H^i_{R_+}(N)$. Let us recall briefly some of those.

(i) If $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ is an exact sequence of finitely generated graded R-modules and homogeneous homomorphisms, then there is a long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0_{R_+}(M,N') \longrightarrow H^0_{R_+}(M,N) \longrightarrow H^0_{R_+}(M,N'') \longrightarrow \dots \\ &\longrightarrow H^i_{R_+}(M,N') \longrightarrow H^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,N'') \longrightarrow \dots \end{aligned}$$

of graded R-modules and homogeneous homomorphisms.

(ii) If N is an R_+-torsion module, then there is a homogeneous isomorphism $H^i_{R_+}(M,N) \longrightarrow Ext^i_R(M,N)$ for all $i \geqslant 0$.

(iii) For all $i \geqslant 0$, the n-th graded component of $H^i_{R_+}(M,N)$, which is denoted by $H^i_{R_+}(M,N)_n$, is a finitely generated R_0-module and it is zero for sufficiently large values of n.

As we mentioned above, our aim in this paper is to extend some of the known results of ordinary graded local cohomology modules to generalized one. Let us describe our purposes precisely.
Throughout the paper,we assume that the base ring R_0 is local with maximal ideal $\mathfrak{m}_0$. In [5], a certain number $g = g(M)$ is defined and

it is proved that if $R = R_0[R_1]$, then $Ass_{R_0}(H^g_{R_+}(M)_n)$ is stable and the R-module $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian for all $i \leqslant g$. In this paper, we introduce an invariant $g(M,N)$ and then we prove a similar result about the R-modules $H^i_{R_+}(M,N)$. In [16], it is proved that if $R = R_0[R_1]$, then $H^d_{R_+}(M)/\mathfrak{m}_0 H^d_{R_+}(M)$ is Artinian. We prove (without the assumption $R = R_0[R_1]$) that if $\mathfrak{q}_0$ is an $\mathfrak{m}_0$-primary ideal, then $H^{l+d}_{R_+}(M,N)/\mathfrak{q}_0 H^{l+d}_{R_+}(M,N)$ is Artinian and R_+-cofinite, where $l = pd(M)$, the projective dimension of M, and $d = dim\frac{N}{\mathfrak{m}_0 N}$. Also, it is shown that $H^{l+d}_{R_+}(M,N)$ is tame. In the second section of the paper we study the asymptotic behaviour of R_0-modules $H^i_{R_+}(M,N)_n$ as n tends to $-\infty$. It is proved, in [6], that $Ass_{R_0} H^r_{R_+}(M)_n$ is stable if $H^i_{R_+}(M)$ is a Noetherian R-module for all $i < r$. We prove that if either $H^i_{R_+}(M,N)$ is Noetherian for all $i < r$ or $H^i_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)})$ is Artinian for all $i < r$, then $Ass_{R_0} H^r_{R_+}(M,N)_n$ is stable. Finally, it is shown that if $g(M,N) < \infty$, then $Ass_{R_0}(H^g_{R_+}(M,N)_n)$ is asymptotically stable.

2. Artinian Properties of Graded Generalized Local Cohomology Modules

Following [5], we define the generalized homological finite length dimension of N with respect to M as

$$g(M,N) = \inf\{j \in \mathbb{N}_0 | l_{R_0} H^j_{R_+}(M,N)_n = \infty \text{ for infinitely many } n \in \mathbb{Z}\}.$$

The following lemma, which is needed in the proof of the next theorem, describes some of the properties of $g(M,N)$.

Lemma 2.1.

(i) $g(M,N) > 0$.

(ii) *If $i < g(M,N)$, then there exists $r \in \mathbb{Z}$ such that $l_{R_0} H^i_{R_+}(M,N)_n < \infty$ for all $n \leqslant r$.*

(iii) *Let $x \in R_+$ be a homogeneous element such that $\Gamma_{R_+}(0 :_N x) = (0 :_N x)$. Then $g(M, \frac{N}{xN}) \geqslant g(M,N) - 1$.*

(iv) *Let x be an indeterminate. Let $R_0' = R_0[x]_{m_0R_0[x]}$, $m_0' = m_0R_0'$, $R' = R_0' \otimes_{R_0} R$, $M' = R_0' \otimes_{R_0} M$ and $N' = R_0' \otimes_{R_0} N$. Then $g(M,N) = g(M',N')$ and, for an Artinian R-module Y, the R'-module $R_0' \otimes_{R_0} Y$ is Artinian.*

(v) $g(M,N) = g(M, \frac{N}{\Gamma_{R_+}(N)})$.

Proof. (i), (ii) and (v) are clear. (iii) Since, in view of the hypothesis, $H^i_{R_+}(M, 0:_N x) \cong Ext^i_R(M, 0:_N x)$ for all $i \geqslant 0$, it is straightforward to see that there exists $r \in \mathbb{Z}$ such that $H^i_{R_+}(M,(0:_N x))_n = 0$ for all $n \leqslant r$ and each $i \leqslant g(M,N)$. Hence if we let $\deg(x) = t$, then, by using the exact sequences $0 \longrightarrow (0:_N x) \longrightarrow N \xrightarrow{x} xN \longrightarrow 0$ and $0 \longrightarrow xN \longrightarrow N \longrightarrow \frac{N}{xN} \longrightarrow 0$ in conjunction with the functorial property of $H^i_{R_+}(M,-)$, we obtain an exact sequence

$$H^{i-1}_{R_+}(M,N)_n \xrightarrow{x} H^{i-1}_{R_+}(M,N)_{n+t} \longrightarrow H^{i-1}_{R_+}(M,\frac{N}{xN})_{n+t} \longrightarrow$$
$$H^{i}_{R_+}(M,N)_n \xrightarrow{x} H^{i}_{R_+}(M,N)_{n+t} \longrightarrow H^{i}_{R_+}(M,\frac{N}{xN})_{n+t}$$

for all $n \leqslant r$ and all $i \leqslant g(M,N)-1$. Now, we can deduce from this, in view of (ii), that $g(M, \frac{N}{xN}) \geqslant g(M,N) - 1$.

(iv) Let $n \in \mathbb{Z}$. Then, in view of [14, 1.9 and 1.6], $H^i_{R_+}(M,N)_n$ is m_0-cofinite if and only if $H^i_{R_+}(M,N)_n$ is an Artinian R_0-module. Since $H^i_{R'_+}(M',N')_n \cong H^i_{R_+}(M,N)_n \otimes_{R_0} R_0'$, it follows that $H^i_{R'_+}(M',N')_n$ is m_0'-cofinite if and only if $H^i_{R_+}(M,N)_n$ is m_0-cofinite. Therefore $H^i_{R_+}(M,N)_n$ is an Artinian R_0-module, if and only if $H^i_{R'_+}(M',N')_n$ is an Artinian R_0'-module. Hence $l_{R_0}(H^i_{R_+}(M,N)_n) < \infty$ if and only if $l_{R_0'}(H^i_{R'_+}(M',N')_n) < \infty$. It now follows from this that $g(M,N) = g(M',N')$. The last part of the lemma follows from [5, Remark 2.1.c]. □

In the following theorem, which is an improvement of [5, 4.2], we use the concept of the Hilbert-Kirby polynomial. Let $X = \oplus_{n\in\mathbb{Z}} X_n$ be a graded Artinian R-module such that, for $n \ll 0$, $l_{R_0}(X_n) < \infty$. Then there is a (uniquely determined) polynomial $P_X \in \mathbb{Q}[x]$ such that $l_{R_0}(X_n) = P_X(n)$ for all $n \ll 0$. P_X is called the Hilbert-Kirby polynomial of X (cf [10]). By convention the zero polynomial has degree -1.

Theorem 2.2. *Let $i \leqslant g(M,N)$. Then $\Gamma_{m_0R}(H^i_{R_+}(M,N))$ is Artinian. Moreover, if $R = R_0[R_1]$, then the Hilbert-Kirby polynomial of $\Gamma_{m_0R}(H^i_{R_+}(M,N))$, denoted by P, is of degree less than i.*

Proof. We prove the theorem by induction on i ($i \geqslant 0$). It is straightforward to see that the result is true when $i = 0$. Suppose, inductively, that $0 < i \leqslant g(M,N)$ and that the result has been proved for $i-1$. By 2.1(v), we have $g := g(M,N) = g(M, \frac{N}{\Gamma_{R_+}(N)})$. Also, it is easy to see that if $\Gamma_{m_0R}(H^i_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)}))$ is Artinian for all $i \leqslant g$, then $\Gamma_{m_0R}(H^i_{R_+}(M,N))$ is Artinian for all $i \leqslant g$. Therefore we may assume that N is R_+-torsion free. Using prime avoidance theorem, we can get a homogeneous N-regular element x of positive degree. Now, we may consider the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ to obtain the exact sequence

$$0 \longrightarrow \frac{H^{i-1}_{R_+}(M,N)}{xH^{i-1}_{R_+}(M,N)} \longrightarrow H^{i-1}_{R_+}(M, \frac{N}{xN}) \longrightarrow (0 :_{H^i_{R_+}(M,N)} x) \longrightarrow 0$$

This sequence, in turn, yields the exact sequence

$$0 \longrightarrow \Gamma_{m_0R}\left(\frac{H^{i-1}_{R_+}(M,N)}{xH^{i-1}_{R_+}(M,N)}\right) \longrightarrow \Gamma_{m_0R}(H^{i-1}_{R_+}(M, \frac{N}{xN}))$$
$$\longrightarrow \Gamma_{m_0R}(0 :_{H^i_{R_+}(M,N)} x) \longrightarrow H^1_{m_0R}\left(\frac{H^{i-1}_{R_+}(M,N)}{xH^{i-1}_{R_+}(M,N)}\right)$$

Therefore, in order to complete the inductive step, it is enough, in view of [13,1.3], to show that $\Gamma_{m_0R}(H^{i-1}_{R_+}(M, \frac{N}{xN}))$ and $H^1_{m_0R}(T)$ are Artinian, where $T = \frac{H^{i-1}_{R_+}(M,N)}{xH^{i-1}_{R_+}(M,N)}$. Since $i \leqslant g$, we see, by 2.1 (iii), that $i-1 \leqslant g-1 \leqslant g(M, \frac{N}{xN})$. Therefore, by the inductive hypothesis, $\Gamma_{m_0R}(H^{i-1}_{R_+}(M, \frac{N}{xN}))$ is Artinian. Next, we have, by [4, 13.1.10], $H^1_{m_0R}(T) = \oplus_{n\in\mathbb{Z}} H^1_{m_0}(T_n)$. Since $i \leqslant g$, the set $A = \{n | l_{R_0}(H^{i-1}_{R_+}(M,N)_n) = \infty\}$ is finite. It therefore follows that $H^1_{m_0R}(T)$ is a direct sum of finitely many R_0-modules $H^1_{m_0}(T_n)$; so that it is an Artinian R-module. For the last part of the theorem, assume that $R = R_0[R_1]$. We use induction on $n = \dim N$ to prove that P is of degree less than i. If $n = 0$, then $H^i_{R_+}(M,N) = Ext^i_R(M,N)$. Hence $\Gamma_{m_0 R}(H^i_{R_+}(M,N))$ is a finitely generated R-module. Therefore $\Gamma_{m_0}(H^i_{R_+}(M,N))_t = 0$ for all $t \ll 0$; and hence $\deg p = -1 < i$. Now, suppose, inductively, that $n > 0$ and that the result has been proved for any finitely generated graded R-module N of dimension less than n. In order to prove the inductive step, we may assume that $\Gamma_{R_+}(N) = 0$ and that

$\frac{R_0}{m_0}$ is an infinite field. Now, there is an element $x \in R_1$ which is a non-zero divisor on N. Using the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$, we obtain the exact sequence $H^{i-1}_{R_+}(M, \frac{N}{xN})_{n+1} \xrightarrow{\psi} H^i_{R_+}(M,N)_n \xrightarrow{x} H^i_{R_+}(M,N)_{n+1}$, which in turn yields the exact sequence

$$0 \longrightarrow \frac{H^{i-1}_{R_+}(M, \frac{N}{xN})_{n+1}}{\ker\psi} \longrightarrow H^i_{R_+}(M,N)_n \xrightarrow{x} H^i_{R_+}(M,N)_{n+1}.$$

Since $i \leqslant g$, it follows that $l_{R_0}(H^{i-1}_{R.0+}(M,N))_n < \infty$ for all $n \ll 0$. Hence $\Gamma_{m_0}(ker\psi) = ker\psi$. Therefore, $\Gamma_{m_0}\left(\frac{H^{i-1}_{R_+}(M,\frac{N}{xN})_{n+1}}{ker\psi}\right) \cong \frac{\Gamma_{m_0}(H^{i-1}_{R_+}(M,\frac{N}{xN}))_{n+1}}{ker\psi}$. One can use the above exact sequence to deduce

$$l_{R_0}(\Gamma_{m_0}(H^i_{R_+}(M,N))_n) \leqslant l_{R_0}(\Gamma_{m_0}(H^i_{R_+}(M,N))_{n+1}) + l_{R_0}\left(\Gamma_{m_0}(H^{i-1}_{R_+}(M,\frac{N}{xN}))_{n+1}\right).$$

Now, we can use this inequality to complete the inductive step.

The concept of tameness, which we will use in the next theorem, is the most fundamental concept related to the asymptotic behaviour of cohomology. A graded R-module $T = \oplus_{n\in\mathbb{Z}} T_n$ is said to be tame (or asymptotically gap-free) if the set $\{n \in \mathbb{Z} | T_n \neq 0, T_{n+1} = 0\}$ is finite. Note that, all graded Artinian R-modules are tame. □

Theorem 2.3. *Assume that* $d = \dim \frac{N}{\mathfrak{m}_0 N}$, $l = pdM$ *and that* $\mathfrak{q}_0$ *is an* $\mathfrak{m}_0$*-primary ideal of* R_0*. Then*

(i) The R-module $\frac{H^{l+d}_{R_+}(M,N)}{\mathfrak{q}_0 H^{l+d}_{R_+}(M,N)}$ *is Artinian and* R_+*-cofinite. Moreover* $H^{l+d}_{R_+}(M,N)$ *is tame.*

(ii) $H^{l+d}_{R_+}(M,N)$ *is Artinian and* R_+*-cofinite whenever* $Supp_{R_0}(N_i) \subseteq \{\mathfrak{m}_0\}$ *for all* $i \in \mathbb{Z}$.

Proof. (i) We prove this by induction on d. If $d = 0$, then $\Gamma_{R_+}(N) = N$ and hence $H^{l+d}_{R_+}(M,N) \cong Ext^l_R(M,N)$. Therefore, since the radical of the annihilator of $\frac{H^l_{R_+}(M,N)}{\mathfrak{q}_0 H^l_{R_+}(M,N)}$ is equal to $\mathfrak{m}_0 + R_+$, the R-module $\frac{H^l_{R_+}(M,N)}{\mathfrak{q}_0 H^l_{R_+}(M,N)}$ is Artinian and R_+-cofinite. Suppose, inductively, that $d > 0$ and the result has been proved for $d-1$. Now, we can use the exact sequence $0 \longrightarrow \Gamma_{R_+}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{R_+}(N)} \longrightarrow 0$, in conjunction with the facts that $H^i_{R_+}(M,\Gamma_{R_+}(N)) \cong Ext^i_R(M,\Gamma_{R_+}(N))$ and $l = pdM$, to see that $H^{l+d}_{R_+}(M,N) \cong H^{l+d}_{R_+}(M,\frac{N}{\Gamma_{R_+}(N)})$. Therefore, since

$\dim \frac{N}{\mathfrak{m}_0 N} = \dim \left(\frac{N}{\Gamma_{R_+}(N)} / \mathfrak{m}_0 \frac{N}{\Gamma_{R_+}(N)}\right)$, we may assume,in addition, that N is R_+-torsion free. As $d > 0$, we also have $R_+ \not\subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \min Ass \frac{N}{\mathfrak{m}_0 N}$. Hence there exists a homogeneous element x of positive degree which avoids all members of $Ass(N)$ and $\min Ass(\frac{N}{\mathfrak{m}_0 N})$. It is straightforward to see that $\dim\left(\frac{N}{xN}/\mathfrak{m}_0 \frac{N}{xN}\right) = d-1$.Hence, by [19, 3.2], $H^{l+d}_{R_+}(M,\frac{N}{xN}) = 0$. Therefore, the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$$

induces the exact sequence

$$H^{l+d-1}_{R_+}(M,\frac{N}{xN}) \longrightarrow H^{l+d}_{R_+}(M,N) \xrightarrow{x} H^{l+d}_{R_+}(M,N) \longrightarrow 0$$

which in turn yields the exact sequence

$$\frac{R_0}{\mathfrak{q}_0} \otimes_{R_0} H^{l+d-1}_{R_+}(M,\frac{N}{xN}) \longrightarrow \frac{R_0}{\mathfrak{q}_0} \otimes_{R_0} H^{l+d}_{R_+}(M,N) \xrightarrow{x} \frac{R_0}{\mathfrak{q}_0} \otimes_{R_0} H^{l+d}_{R_+}(M,N) \longrightarrow 0.$$

Now, one can use the above exact sequence in conjunction with the inductive hypothesis to see that the R-module $0 :_{\frac{H^{l+d}_{R_+}(M,N)}{\mathfrak{q}_0 H^{l+d}_{R_+}(M,N)}} x$ is Artinian and R_+-cofinite. Therefore,in view of [15, 4.1], the inductive step is completed and the result follows by induction. In particular,the R-module $\frac{H^{l+d}_{R_+}(M,N)}{\mathfrak{m}_0 H^{l+d}_{R_+}(M,N)}$ is Artinian; so that it is tame. It therefore follows, in view of Nakayama's Lemma, that $H^{l+d}_{R_+}(M,N)$ is tame.

(ii) We argue by induction on d. If $d=0$ then $N=\Gamma_{R_+}(N)$. Hence in view of the hypothesis, N is Artinian. Therefore, since $H^l_{R_+}(M,N) \cong Ext^l_{R_+}(M,N)$, we see that $H^l_{R_+}(M,N)$ is Artinian and R_+-cofinite. Suppose, inductively, that $d>0$ and the result has been proved for smaller values of d. Since $d>0$, it follows that $H^{l+d}_{R_+}(M,N) \cong H^{l+d}_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)})$. So we may assume that N is R_+-torsion free. Now, as in the proof of (i), there exists a homogeneous N- regular element x of positive degree which is not belong to any minimal element of $Ass\frac{N}{\mathfrak{m}_0 N}$. Therefore $\dim \frac{\frac{N}{xN}}{\mathfrak{m}_0(\frac{N}{xN})} \leqslant d-1$. Now, we may use the exact sequence

$$H^{l+d-1}_{R_+}(M, \frac{N}{xN}) \longrightarrow H^{l+d}_{R_+}(M,N) \xrightarrow{x} H^{l+d}_{R_+}(M,N) \longrightarrow 0$$

in conjunction with the inductive hypothesis and [15, 4.1] to see that $H^{l+d}_{R_+}(M,N)$ is Artinian and R_+-cofinite. $\square$

Proposition 2.4. *$H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ is Artinian for all $i \geqslant 0$. Moreover, if $R=R_0[R_1]$, then the Hilbert-Kirby polynomial of $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ is of degree less than i.*

Proof. By [9, 4.2], $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ is Artinian. Now, we prove the last part of the proposition by induction on $t=\dim(\Gamma_{\mathfrak{m}_0 R}(N))$. Let p_i be the Hilbert-Kirby polynomial of $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$. If $t=0$, then it is immediate to see that $\deg(p_i)=-1<i$. Suppose that $t>0$ and that the result has been proved for smaller values of t. Put $\Gamma_{\mathfrak{m}_0 R}(N)=N'$. We may assume that $\frac{R_0}{\mathfrak{m}_0}$ is an infinite field. Then there exists $x \in R_1$ such that $\Gamma_{R_+}(0 :_{N'} x) = (0 :_{N'} x)$. Now, it is straightforward to see that $H^i_{R_+}(M, (0 :_{N'} x))_n = 0$ for all $n \ll 0$. Therefore, we can use the exact sequences $0 \longrightarrow (0 :_{N'} x) \longrightarrow N' \longrightarrow xN' \longrightarrow 0$ and $0 \longrightarrow xN' \longrightarrow N' \longrightarrow \frac{N'}{xN'} \longrightarrow 0$ to obtain an exact sequence

$$H^{i-1}_{R_+}(M, \frac{N'}{xN'})_{n+1} \longrightarrow H^i_{R_+}(M,N')_n \xrightarrow{x} H^i_{R_+}(M,N')_{n+1}.$$

This exact sequence yields the inequality

$$l_{R_0} H^i_{R_+}(M, N')_n \leqslant l_{R_0} H^i_{R_+}(M, N')_{n+1} + l_{R_0} H^{i-1}_{R_+}(M, \frac{N'}{xN'})_{n+1}.$$

As $\dim \frac{N'}{xN'} < t$, we can use the above inequality to conclude the result by induction. □

Corollary 2.5. *Let $\bar{N} = \frac{N}{\Gamma_{\mathfrak{m}_0 R}(N)}$, $\bar{d} = \dim \frac{\bar{N}}{\mathfrak{m}_0 \bar{N}}$, $pdM = l$ and let $\mathfrak{q}_0$ be an $\mathfrak{m}_0$-primary. Then*

(a) $H^i_{R_+}(M, N)$ is Artinian for all $i > \bar{d} + l$.

(b) $\frac{H^{l+\bar{d}}(M,N)}{\mathfrak{q}_0 H^{l+\bar{d}}_{R_+}(M,N)}$ is Artinian and $H^i_{R_+}(M, N)$ is tame for all $i \geqslant \bar{d}+l$.

Proof.

(a): The exact sequence $0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{\mathfrak{m}_0 R}(N)} \longrightarrow 0$ induces an exact sequence

$$H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N)) \xrightarrow{\lambda} H^i_{R_+}(M, N) \xrightarrow{\psi} H^i_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0 R}(N)}) \xrightarrow{\varphi} H^{i+1}_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N)).$$

Since $H^i_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0 R}(N)}) = 0$ for all $i \geqslant l + \bar{d} + 1$, we can use the above exact sequence and (2.4) to conclude (a).

(b): Again we can use the above exact sequence in conjunction with (2.3) and (2.4) to see that, $im\lambda, im\varphi$ and $Tor^{R_0}_1(im\varphi, \frac{R_0}{\mathfrak{q}_0})$ are all Artinian. It therefore follows that $H^{l+\bar{d}}_{R_+}(M, N) \otimes_{R_0} \frac{R_0}{\mathfrak{q}_0}$ is Artinian. In particular, $\frac{H^{l+\bar{d}}_{R_+}(M,N)}{\mathfrak{m}_0 H^{l+\bar{d}}_{R_+}(M,N)}$ is Artinian; and therefore it is tame. Now, we can use Nakayama's Lemma to see that $H^{l+\bar{d}}_{R_+}(M, N)$ is tame. Since $H^i_{R_+}(M, N)$ is Artinian for all $i > l + \bar{d}$, the result follows. □

In 2.3(ii), under certain conditions,we proved that the R-module $H^{l+d}_{R_+}(M, N)$ is Artinian and R_+-cofinite. However, in non graded case, the following theorem holds.

Theorem 2.6. *Let $(A, \mathfrak{m})$be a local Noetherian ring and let M and N be finitely generated A-modules. Suppose that $n = \dim N$, $pdM = l$ and* *that $\mathfrak{a}$ is an ideal of A. Then the A-module $H^{n+l}_{\mathfrak{a}}(M,N)$ is Artinian and $\mathfrak{a}$-cofinite. Furthermore the set $Ass_A H^{n+l}_{\mathfrak{a}}(M,N)$ is finite.*

Proof. We prove by induction on $n = \dim N$. If $\dim N = 0$. then N is $\mathfrak{m}$-torsion, and hence $\mathfrak{a}$-torsion module. Therefore $H^l_{\mathfrak{a}}(M, \Gamma_{\mathfrak{m}}(N)) = Ext^l_A(M, \Gamma_{\mathfrak{m}}(N))$ is a finitely generated A-module. Hence $H^l_{\mathfrak{a}}(M,N)$ is an $\mathfrak{a}$-cofinite and Artinian A-module.

Now suppose, inductively, that $\dim N = n > 0$ and the result has been proved for all finitely generated A-modules of dimension smaller than n. Consider the exact sequence $0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{\mathfrak{a}}(N)} \longrightarrow 0$ and note that $H^{l+i}_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) = Ext^{l+i}_R(M, \Gamma_{\mathfrak{a}}(N)) = 0$ for all $i > 0$. Therefore $H^{l+n}_{\mathfrak{a}}(M,N) \cong H^{l+n}_{\mathfrak{a}}(M, \frac{N}{\Gamma_{\mathfrak{a}}(N)})$. Thus we may assume that $\Gamma_{\mathfrak{a}}(N) = 0$. Then there exists $x \in \mathfrak{a}$ such that x is an an N-sequence. It is straight forward to see that $H^{l+n}_{\mathfrak{a}}(M, \frac{N}{xN}) = 0$. Therefore, we can use the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ to obtain the exact sequence $H^{l+n-1}_{\mathfrak{a}}(M, \frac{N}{xN}) \longrightarrow H^{l+n}_{\mathfrak{a}}(M,N) \xrightarrow{x} H^{l+n}_{\mathfrak{a}}(M,N) \longrightarrow 0$. Using this exact sequence, one can deduce, by the inductive hypothesis and [13,1.3], that the A-module $H^{l+n}_{\mathfrak{a}}(M,N)$ is Artinian and $\mathfrak{a}$-cofinite. Therefore $Ass_A H^{l+n}_{\mathfrak{a}}(M,N)$ is a finite set by [14,1.4]. □

3. Asymptotic Behaviour of Graded Components of Graded Generalized Local Cohomology Modules

In this section, we study the asymptotic stability of the set of associated primes of certain generalized local cohomology modules,that is the question whether, for a fixed integer i,the set of associated primes $Ass_{R_0} H^i_{R_+}(M,N)_n$ of the R_0- module $H^i_{R_+}(M,N)_n$ becomes ultimately constant if n tends to $-\infty$.It is easy to see that

$$Ass_R H^i_{R_+}(M,N) =$$
$$\{\mathfrak{p} \in Spec(R) | \mathfrak{p} \cap R_0 + R_+ = \mathfrak{p}, \mathfrak{p} \cap R_0 \in Ass_{R_0} H^i_{R_+}(M,N)_n \text{ for some } n \in \mathbb{Z}\}$$

Therefore $Ass_R H^i_{R_+}(M,N)$ is a finite set whenever $Ass_{R_0} H^i_{R_+}(M,N)_n$ is asymptotically stable for $n \longrightarrow -\infty$. The finiteness dimension of M

and N relative to R_+ is defined by

$$f = f(M,N) = \inf\{j \in \mathbb{N}_0 | H^j_{R_+}(M,N) \text{ is not finitely generated}\}$$

and it is proved,in [9, 3.5],that if $R = R_0[R_1]$ then $(Ass_{R_0} H^f_{R_+}(M,N)_n)_{n\in\mathbb{Z}}$ is asymptotically stable.Since $f(M,N) \leqslant g(M,N)$,the next theorem provides a generalization of the above mentioned result. At this stage the following remark is needed.

Remark 3.1. *Let $f : R_0 \longrightarrow R'_0$ be a faithful flat ring homomorphism and let $R' = R'_0 \otimes_{R_0} R$, $M' = R'_0 \otimes_{R_0} M$ and $N' = R'_0 \otimes_{R_0} N$. Then $H^K_{R'_+}(M',N')_n \cong H^K_R(M,N)_n \otimes_{R_0} R'_0$. Note that $H^K_{R_+}(M,N)$ is Artinian (resp. Noetherian) if and only if $H^K_{R'_+}(M',N')$ is Artinian (resp. Noetherian). Moreover*

$$Ass_{R_0} H^i_{R_+}(M,N)_n = \{\mathfrak{p}'_0 \cap R_0 | \mathfrak{p}'_0 \in Ass_{R'_0} H^i_{R'_+}(M',N')_n\}$$

for all $n \in \mathbb{Z}$ [12, 23.2]. It follows that $Ass_{R_0} H^i_{R_+}(M,N)_n)_{n\in\mathbb{Z}}$ is asymptotically stable if and only if $(Ass_{R'_0} H^i_{R'_+}(M',N')_n)_{n\in\mathbb{Z}}$ is asymptotically stable.

Theorem 3.2. *Let $R = R_0[R_1]$ and $g(M,N) < \infty$.*
Then $(Ass_{R_0} H^g_{R_+}(M,N)_n)_{n\in\mathbb{Z}}$ is asymptotically stable.

Proof. If $f(M,N) = g(M,N)$, then the result is clear by [9,3.5]. So, let $f < g$. Then $Ass_{R_0} H^f_{R_+}(M,N)_n = \{m_0\}$ for all $n \ll 0$. Hence $Ass_{R_0} H^g_{R_+}(M,N)_n - \{m_0\} \neq \phi$ for all $n \ll 0$.

Let $(\hat{R}_0, \hat{m}_0)$ denote the m_0-adic completion of the local ring (R_0, m_0) and let $M \otimes_{R_0} \hat{R}_0 = \hat{M}$ and $N \otimes_{R_0} \hat{R}_0 = \hat{N}$. In view of 3.2 we may assume that R_0 is complete. Then the set $A = \bigcup_{n\in\mathbb{Z}} Ass_{R_0} H^g_{R_+}(M,N)_n - \{m_0\}$is not empty. Hence, by [11,3.2], there exists $x \in m_0$ such that $x \notin \bigcup_{\mathfrak{p}\in A} \mathfrak{p}$. Let $S = \{x^K | 0 \leqslant K \in \mathbb{Z}\}$. Then $S \cap m_0 \neq \phi$. Hence $f(S^{-1}M, S^{-1}N) \leqslant g(M,N)$. If $i < g(M,N)$ then $l_{R_0} H^i_{R_+}(M,N)_n < \infty$ for all $n \ll 0$. Therefore there exists $t \in \mathbb{Z}_0$ such that $m_0^t H^i_{R_+}(M,N)_n = 0$ for all $n \ll 0$. Hence $S^{-1} H^i_{R_+}(M,N)_n = 0$ for all $n \ll 0$. It follows that $g(M,N) = f(S^{-1}M, S^{-1}N)$.

Therefore $(Ass_{S^{-1}R_0}(H^g_{S^{-1}R_+}(S^{-1}M, S^{-1}N)_n))_{n\in\mathbb{Z}}$ is stable.
Hence there exists $n_0 \in \mathbb{Z}$ such that $Ass_{S^{-1}R_0}(H^g_{S^{-1}R_+}(S^{-1}M, S^{-1}N)_n) = Ass_{S^{-1}R_0}(H^g_{S^{-1}R_+}(S^{-1}M, S^{-1}N)_{n_0})$ for all $n \ll n_0$.
Thus $Ass_{R_0}H^g_{R_+}(M,N)_n - \{m_0\} = Ass_{R_0}H^g_{R_+}(M,N)_{n_0}$ for all $n \ll n_0$.
Note that, by 2.2,$\Gamma_{m_0R}(H^g_{R_+}(M,N))$ is an Artinian R-module. Hence $\Gamma_{m_0R}(H^g_{R_+}(M,N))$ is tame and $\Gamma_{m_0}(H^g_{R_+}(M,N)_n)$ is an Artinian R_0-module for all n.
Thus $Supp_{R_0}\Gamma_{m_0}(H^g_{R_+}(M,N)_n) subs \{m_0\}$ for all $n \in \mathbb{Z}$.
Now, using the exact sequence

$$0 \longrightarrow \Gamma_{m_0}(H^g_{R_+}(M,N)_n) \longrightarrow H^g_{R_+}(M,N)_n \longrightarrow \frac{H^g_{R_+}(M,N)_n}{\Gamma_{m_0}(H^g_{R_+}(M,N)_n)} \longrightarrow 0,$$

we get $Ass_{R_0}H^g_{R_+}(M,N)_n = Ass_{R_0}H^g_{R_+}(M,N)_{n_0}$ for all $n \ll n_0$. □

Now, we state a Lemma which will be used in the remaining part of the paper.

Lemma 3.3. ([3]) *Let $T = \oplus_{n\in\mathbb{Z}}T_n$ be a graded R-module such that $\frac{T}{m_0T}$ is Artinian and that T_n is a finitely generated R_0-module for all $n \in \mathbb{Z}$. Then, there exists $t \in \mathbb{Z} \cup \{\infty\}$ such that, for each $x \in R_1 - \bigcup_{\mathfrak{p}\in Att\frac{T}{\mathfrak{m}_0T} - v(R_+)} \mathfrak{p}$ and all $n < t$, the multiplication map $T_n \xrightarrow{x} T_{n+1}$ is surjective.*

Theorem 3.4. *Let $R = R_0[R_1]$ and suppose that $H^i_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)})$ is Artinian for all $i < r$. Then $(Ass_{R_0}H^r_{R_+}(M,N)_n)_{n\in\mathbb{Z}}$ is asymptotically stable.*

Proof. As $H^0_{R_+}(M,N)_n = (Hom(M, \Gamma_{R_+}(N))_n = 0$ for all $n \ll 0$, the case where $r = 0$ is clear. So, let $r > 0$.

It is straightforward to see that $(Ass_{R_0}H^r_{R_+}(M,N)_n)_{n\in\mathbb{Z}}$ is stable if and only if $(Ass_{R_0}H^r_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)})_n)_{n\in\mathbb{Z}}$ is stable. Therefore, we may assume that $\Gamma_{R_+}(N) = 0$. Also, in the view of 3.2 and 2.1(iv), we may assume, in addition, that $\frac{R_0}{m_0}$ is an infinite field. Put $A = Ass(N) \cup \cup_{i<r} Att\frac{H^i_{R_+}(M,N)}{m_0H^i_{R_+}(M,N)} - V(R_+)$. Then, by [7, 1.5.12], there exists a homogeneous N-regular element $x \in R_1 - \bigcup_{\mathfrak{p}\in A}\mathfrak{p}$.

Now, we consider the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow$

0 to deduce the exact sequence $H^{i-1}_{R_+}(M,N) \longrightarrow H^{i-1}_{R_+}(M,\frac{N}{xN}) \longrightarrow H^i_{R_+}(M,N) \xrightarrow{x} H^i_{R_+}(M,N)$. Using this exact sequence we see that $H^i_{R_+}(M,\frac{N}{xN})$ is Artinian for all $i < r-1$. Therefore, by 3.1, the above exact sequence, yields the exact sequence

$$0 \longrightarrow H^{i-1}_{R_+}(M,\frac{N}{xN})_{n+1} \longrightarrow H^i_{R_+}(M,N)_n \xrightarrow{x} H^i_{R_+}(M,N)_{n+1}$$

for all $n \ll 0$. Hence

$$\begin{aligned} Ass_{R_0} H^{i-1}_{R_+}(M,\frac{N}{xN})_{n+1} &\subseteq Ass_{R_0} H^i_{R_+}(M,N)_n \\ &\subseteq Ass_{R_0} H^{i-1}_{R_+}(M,\frac{N}{xN})_{n+1} \cup Ass_{R_0} H^i_{R_+}(M,N)_{n+1}. \end{aligned}$$

Now, one can deduce ,by induction on r, that $(Ass_{R_0} H^r_{R_+}(M,N)_n)_{n\in\mathbb{Z}}$ is asymptotically stable for $n \longrightarrow -\infty$. $\square$

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