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On a Question of Allen Shields

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Abstract. We give sufficient conditions for the boundedness of the analytic projection on the set of multipliers of the Banach weighted Hardy spaces. This presents the sufficient conditions to a problem that has considered by A. L. Shields.

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1. Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 < p < \infty$, the space $L^p(\beta)$ consists of all formal Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$$

such that the norm

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$$

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is finite. When n just runs over $\mathbb{N} \cup \{0\}$, the space $L^p(\beta)$ only contains formal power series

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

and it is usually denoted by $H^p(\beta)$. If $p = 2$, such spaces were introduced by Allen L. Shields to study weighted shift operators in the paper [4] which is one of basic works in this area and is a pretty large work that contains a number of interesting results, and indeed it is mainly of auxiliary nature. Actually, Shields showed a close relation between injective weighted shifts and the multiplication operator M_z acting on $L^2(\beta)$ or $H^2(\beta)$ (see [4, Proposition 7]). These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k\in\mathbb{Z}}$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. Clearly M_z, the operator of multiplication by z on $L^p(\beta)$, shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case

$$\|M_z^n\| = \sup_k [\beta(k+n)/\beta(k)] \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

We say that a complex number λ is a bounded point evaluation on $L^p(\beta)$ if the functional $e(\lambda) : L^p(\beta) \longrightarrow \mathbb{C}$ defined by $e(\lambda)(f) = f(\lambda)$ is bounded.

Let X be a Banach space. It is convenient and helpful to introduce the notation $< x, x^* >$ to stand for $x^*(x)$, for $x \in X$ and $x^* \in X^*$. Also, the set of bounded linear operators on X is denoted by $B(X)$. If $A \in B(X)$, by $\sigma(A)$ we mean the spectrum of A and by $r(A)$ we mean the spectral radius of A.

By the same method used in [7], we can see that $(L^p(\beta))^* = L^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p} + \frac{1}{q} = 1$. Also, if

$$f(z) = \sum_n \hat{f}(n) z^n \in L^p(\beta)$$

and

$$g(z) = \sum_n \hat{g}(n) z^n \in L^q(\beta^{\frac{p}{q}}),$$

then clearly

$$< f, g >= \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p.$$

For a good source in special case of formal power series see [1,3–17].

If Ω is a domain in the complex plane $\mathbb{C}$, then by $H(\Omega)$ and $H^\infty(\Omega)$ we mean respectively the set of analytic functions and the set of bounded analytic functions on Ω. By $\|\cdot\|_\Omega$ we denote the supremum norm on Ω.

We denote the set of multipliers

$$\{\varphi \in L^p(\beta) : \varphi L^p(\beta) \subseteq L^p(\beta)\}$$

by $L^p_\infty(\beta)$ and the linear transformation of multiplication by φ on $L^p(\beta)$ by M_φ. The space $L^p_\infty(\beta)$ is a commutative Banach algebra with the norm $\|\varphi\|_\infty = \|M_\varphi\|$.

By an analytic projection we mean the map which send each two-sided sequence $\{a_n\}_{-\infty}^{\infty}$ into the corresponding one-sided sequence $\{a_n\}_0^\infty$ ([4]).

It is known that the analytic projection for the unweighted shift is unbounded ([1, Prob. 9, Chapter 14]). In this paper, we want to investigate conditions under which a bilateral shift is an analytic projection on $L^p_\infty(\beta)$.

2. Boundedness of Analytic Projection

First we note that the multiplication operator M_z on $L^p(\beta)$ is unitarily equivalent to an injective bilateral weighted shift and conversely, every injective bilateral weighted shift is unitarily equivalent to M_z acting on $L^p(\beta)$ for a suitable choice of β. Throughout this paper, M_z is a bounded operator on $L^p(\beta)$. Let

$$\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k = \varphi_1(z) + \varphi_2(z)$$

be in $L^p_\infty(\beta)$ where

$$\begin{aligned}\varphi_1(z) &= \sum_{k=0}^{\infty}\hat{\varphi}(k)z^k,\\ \varphi_2(z) &= \sum_{k=1}^{\infty}\hat{\varphi}(-k)z^{-k}.\end{aligned}$$

Define the analytic projection $J : L^p_\infty(\beta) \longrightarrow L^p(\beta)$ by $J(\varphi) = \varphi_1$. Then the projection J is clearly bounded since

$$\|J(\varphi)\|_p = \|\varphi_1\|_p \leqslant \|\varphi\|_p \leqslant \|\varphi\|_\infty.$$

The problem of boundedness raised when we consider the projection J from $L^p_\infty(\beta)$ into $L^p_\infty(\beta)$. It is certainly bounded if M_z is not invertible on $L^p(\beta)$, since in this case we will see that for

$$\varphi(z) = \sum_{k=-\infty}^{\infty}\hat{\varphi}(k)z^k$$

in $L^p_\infty(\beta)$, it should be $\hat{\varphi}(n) = 0$ for all $n < 0$. But this projection is not necessarily a bounded operator on $L^p_\infty(\beta)$. As stated in [4], it is not bounded when $\beta(n) = 1$ for all n in $\mathbb{Z}$ ([18, Chapter VII, equations (2.2) and (2.3)]). We want to investigate that for which multiplication operators the analytic projection J is a bounded linear operator from $L^p_\infty(\beta)$ into $L^p_\infty(\beta)$. This present the sufficient conditions to the following question that has considered by Shields in [4, page 91, Question 11]:

Question 2.1. For which bilateral shifts is the analytic projection a bounded operator on $L^2_\infty(\beta)$?

We will use the following notations:

$$\begin{aligned}
r_{01} = \overline{\lim}\beta(-n)^{\frac{-1}{n}} \quad &, \quad \Omega_{01} = \{z \in \mathbb{C} : |z| > r_{01}\}\\
r_{11} = \underline{\lim}\beta(n)^{\frac{1}{n}} \quad &, \quad \Omega_{11} = \{z \in \mathbb{C} : |z| < r_{11}\}\\
r_{12} = r(M_z^{-1})^{-1} \quad &, \quad \Omega_{12} = \{z \in \mathbb{C} : |z| > r_{12}\}\\
r_{22} = r(M_z) \quad &, \quad \Omega_{22} = \{z \in \mathbb{C} : |z| < r_{22}\}\\
\Omega_1 = \Omega_{01} \cap \Omega_{11} \quad &= \quad \{z \in \mathbb{C} : r_{01} < |z| < r_{11}\}\\
\Omega_2 = \Omega_{12} \cap \Omega_{22} \quad &= \quad \{z \in \mathbb{C} : r_{12} < |z| < r_{22}\}.
\end{aligned}$$

If $r_{01} < r_{11}$, then by the same method used for the formal power series in [7] we can see that each point of Ω_1 is a bounded point evaluation on $L^p(\beta)$.
Note that for the algebra $B(X)$ of all bounded operators on a Banach space X, the weak operator topology is the one such that $A_\alpha \longrightarrow A$ in the weak operator topology if and only if $A_\alpha x \longrightarrow Ax$ weakly, $x \in X$.

Definition 2.2. *Let X be a Banach space. A compact subset K of the plane is a spectral set of $A \in B(X)$, if it contains the spectrum of A, $\sigma(A)$, and*

$$\|f(A)\| \leqslant \max\{|f(z)| : z \in K\}$$

for all rational functions f with poles off K.
From now on we assume that M_z is bounded on $L^p(\beta)$ and $r_{01} < r_{11}$.

Theorem 2.3. *Let M_z be invertible on $L^p(\beta)$, $r_{12} < r_{22}$ and $\sigma(M_z)$ be a spectral set for M_z. Then $J \in B(L^p_\infty(\beta))$.*

Proof. Note that $\sigma(M_z) = \bar{\Omega}_2$. Since $\sigma(M_z)$ is a spectral set, we have $\|M_p\| \leqslant c\|p\|_{\Omega_2}$ for all polynomials p. Now let $q(z) = q_1(z) + q_2(z)$ be a Laurent polynomial such that

$$\begin{aligned} q_1(z) &= \sum_{k=0}^{n} \hat{q}(k) z^k, \\ q_2(z) &= \sum_{k=1}^{n} \hat{q}(-k) z^{-k}. \end{aligned}$$

Thus by Lemma in [3, p.81], we have

$$\|J(q)\|_{\Omega_{22}} = \|q_1\|_{\Omega_{22}} \leqslant c_1 \|q\|_{\Omega_2},$$

where

$$c_1 = 1 + r_{12}(r_{22}^2 - r_{12}^2)^{\frac{1}{2}}.$$

Therefore we get

$$\|M_{q_1}\| \leqslant c\|q_1\|_{\Omega_2} \leqslant c\|q_1\|_{\Omega_{22}} \leqslant cc_1\|q\|_{\Omega_2}.$$

But $L^p_\infty(\beta) \subset H(\Omega_2)$ and each point of Ω_2 is a bounded point evaluation on $L^p_\infty(\beta)$, because if

$$\varphi = \sum_n \hat{\varphi}(n) z^n \in L^p_\infty(\beta),$$

then the relation

$$< M_\varphi f_m, f_n >= \hat{\varphi}(n-m)\beta(n)^p$$

implies that

$$\begin{aligned} |\hat{\varphi}(n-m)|\beta(n)^p &\leqslant \|M_\varphi\| \|f_m\|_{L^P(\beta)} \cdot \|f_n\|_{L^q(\beta^{\frac{p}{q}})} \\ &= \|M_\varphi\| \beta(m)\beta(n)^{\frac{p}{q}}. \end{aligned}$$

By taking $k = n - m$, we get

$$|\hat{\varphi}(k)| \leqslant \|M_\varphi\| \frac{\beta(m)}{\beta(m+k)}$$

for all m. Therefore,

$$|\hat{\varphi}(k)| \leqslant \|M_\varphi\| \|M_z^k\|^{-1} \qquad (**)$$

and so

$$|\varphi(z)| \leqslant \|M_\varphi\| \sum_n \frac{|z|^n}{\|M_z^n\|},$$

where by the Root test the series $\sum_n |z|^n \{\|M_z^n\|\}$ converges on Ω_2. This implies that $L^p_\infty(\beta) \subset H(\Omega_2)$ and each points of Ω_2 is a bounded point evaluation on $L^p_\infty(\beta)$. Indeed Ω_2 is the largest open annulus such that $L^p_\infty(\beta) \subset H^\infty(\Omega_2)$ (see [4] for the case of formal power series). Since $L^p_\infty(\beta)$ is a commutative Banach algebra and e_λ is multiplicative, it should be $\|e_\lambda\| = 1$ for all $\lambda \in \Omega_2$, and this implies that $\|\psi\|_{\Omega_2} \leqslant \|M_\psi\|$ for all ψ in $L^p_\infty(\beta)$. Thus

$$\|M_{J(q)}\| \leqslant cc_1 \|M_q\|$$

for all Laurent polynomials q. This implies that $J \in B(L^p_\infty(\beta))$ and so the proof is complete. $\square$

Theorem 2.4. *Let M_z be invertible on $L^p(\beta)$, $r_{12} < r_{22}$ and $\|M_p\| \leqslant c\|p\|_{\Omega_2}$ for all polynomials p in z. Then $J \in B(L^p_\infty(\beta))$.*

Proof. In the proof of Theorem 2.3, we only used the inequality in the definition of the spectral set for polynomials instead of rational functions. So Theorem 2.3, is also consistent if we substitute the relation " $\|M_p\| \leqslant c\|p\|_{\Omega_2}$ for all polynomials p in z ", instead of the relation "$\sigma(T)$ is a spectral set". This completes the proof. $\square$

Theorem 2.5. *Let M_z be invertible on $L^p(\beta)$, $r_{12} < r_{22}$ and $\|M_q\| \leqslant c\|q\|_{\Omega_2}$ for all polynomials q in z^{-1}. Then $J \in B(L^p_\infty(\beta))$.*

Proof. Put $J_1 = I - J$. Then for a Laurent polynomial

$$p(z) = \sum_{k=-n}^{n} \hat{p}(k) z^k = p_1(z) + p_2(z),$$

where

$$p_1(z) = \sum_{k=0}^{n} \hat{p}(k) z^k$$

and

$$p_2(z) = \sum_{k=1}^{n} \hat{p}(-k) z^{-k},$$

we have $J_1(p) = p_2$. Now by Lemma in [3, p.81] we get:

$$\|p_2\|_{\Omega_{12}} = \|J_1(p)\|_{\Omega_{12}} \leqslant c_1 \|p\|_{\Omega_2} \leqslant c_1 \|M_p\|$$

where,

$$c_1 = 1 + r_{12}(r_{22}^2 - r_{12}^2)^{\frac{1}{2}}.$$

Thus by the hypothesis

$$\begin{aligned} \|M_{J_1(p)}\| &= \|M_{p_2}\| \leqslant c\|p_2\|_{\Omega_2} \\ &\leqslant c\|p_2\|_{\Omega_{12}} \leqslant cc_1\|M_p\|. \end{aligned}$$

This implies that $J \in B(L^p_\infty(\beta))$ and so the proof is complete. $\square$

Proposition 2.6. *If M_z is not invertible on $L^p(\beta)$, then $J \in B(L^p_\infty(\beta))$.*

Proof. Let $\varphi \in L^p_\infty(\beta)$. Since M_z is not invertible, $\inf_m \frac{\beta(m+1)}{\beta(m)} = 0$. Now by the relation $(**)$ in the proof of Theorem 2.3, we have

$$\begin{aligned} |\hat{\varphi}(k)| &\leqslant \|M_\varphi\| \|M_z^k\|^{-1} \\ &= \|M_\varphi\| \inf_m \frac{\beta(m)}{\beta(m+k)} \end{aligned}$$

for all $k \in \mathbb{Z}$. So for $k = -1$ we get $\hat{\varphi}(-1) = 0$ for all $\varphi \in L^p_\infty(\beta)$. Multiplying by z we have $\hat{\varphi}(-2) = \hat{z\varphi}(-1) = 0$, etc. Thus $\hat{\varphi}(n) = 0$ for all $n \leqslant -1$ and so J is identity, i.e., $J(\varphi)(z) = \varphi(z)$ for all $\varphi \in L^p_\infty(\beta)$ and so the proof is complete. $\square$

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