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Reflexivity on Banach Spaces of Analytic Functions

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"Dedicated to Mola Ali"

Abstract. Let $\mathcal{X}$ be a Banach space of functions analytic on a plane domain Ω such that for every λ in Ω the functional of evaluation at λ is bounded. Assume further that $\mathcal{X}$ contains the constants and admits multiplication by the independent variable z, M_z, as a bounded operator. We give sufficient conditions for M_z to be reflexive.

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1. Introduction

Let $\mathcal{X}$ be a separable reflexive Banach space whose elements are analytic functions on a complex domain Ω. It is convenient and helpful to introduce the notation $< x, x^* >$ to stand for $x^*(x)$, for $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. Assume $1 \in \mathcal{X}$ and the operator M_z of multiplication by z maps $\mathcal{X}$ into itself and for each λ in Ω, the functional $e(\lambda) : \mathcal{X} \to C$ of evaluation at λ given by

$$e(\lambda)(f) =< f, e(\lambda) >= f(\lambda).$$

is bounded.

For the algebra $\mathcal{B}(\mathcal{X})$ of all bounded operators on a Banach space $\mathcal{X}$, the weak operator topology is the one in which a net A_α converges to A if $A_\alpha x \to Ax$ weakly, $x \in \mathcal{X}$ ([7]).

A complex valued function φ on Ω for which $\varphi f \in \mathcal{X}$ for every $f \in \mathcal{X}$ is called a multiplier of $\mathcal{X}$ and the collection of all these multipliers is denoted by $\mathcal{M}(\mathcal{X})$. Because M_z is a bounded operator on $\mathcal{X}$, the adjoint $M_z^* : \mathcal{X}^* \to \mathcal{X}^*$ satisfies

$$M_z^* e(\lambda) = \lambda e(\lambda).$$

In general each multiplier φ of $\mathcal{X}$ determines a multiplication operator M_φ defined by $M_\varphi f = \varphi f, f \in \mathcal{X}$. Also

$$M_\varphi^* e(\lambda) = \varphi(\lambda) e(\lambda).$$

It is well-known that each multiplier is a bounded analytic function. Indeed $|\varphi(\lambda)| \leqslant ||M_\varphi||$ for each λ in Ω. Also $M_\varphi 1 = \varphi \in \mathcal{X}$. But $\mathcal{X} \subset H(\Omega)$, thus φ is a bounded analytic function.

Recall that if $A \in \mathcal{B}(\mathcal{X})$, then $Lat(A)$ is by definition the lattice of all invariant subspaces of A, and $AlgLat(A)$ is the algebra of all operators

B in $\mathcal{B}(\mathcal{X})$ such that $Lat(A) \subset Lat(B)$. An operator A in $\mathcal{B}(\mathcal{X})$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathcal{B}(\mathcal{X})$ that contains A and the identity I and is closed in the weak operator topology.

For G an open connected (not necessarily simply connected) subset of the complex plane and α an ordinal number, the set G_α is defined as in Sarason [4, p.525]. Here we only remember the definition of the Caratheodory hull. By a domain we understand a connected open subset of the plane. If B is a bounded domain in the plane, then the Caratheodory hull (or $\mathbb{C}$-hull) of B is the complement of the closure of the unbounded component of the complement of the closure of B. The $\mathbb{C}$-hull of B is denoted by B^*. Intuitively, B^* can be described as the interior of the outer boundary of B, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that

$$|p(z_0)| \leqslant \sup\{|p(z)| : z \in B\},$$

for all polynomials p. The components of B^* are simply connected; in fact, one can easily see that each of these components has a connected complement. The component of B^* that contains B is denoted by B_1. Note that for all polynomials p, $\|p\|_B = \|p\|_{B_1}$.

2. Main Result

The operator M_z has been the focus of attention for several decades and many of its properties have been studied (e.g. [1],[6]). In this article we would like to give some sufficient conditions so that the operator M_z becomes reflexive (for a good source of reflexivity see [3]). This is a continuation of our work [5] where we only considered finitely connected domains, but here we work with arbitrary domains.

Theorem. *Let $\mathcal{X}$ be a separable reflexive Banach space whose elements are analytic functions on a complex domain Ω, each point of which is a bounded point evaluation. Suppose that $\mathcal{X}$ contains the constant functions and $z \in \mathcal{M}(\mathcal{X})$. If $\{e(\lambda) : \lambda \in \Omega\}$ is norm bounded and $H^\infty(\Omega_1) \subset \mathcal{M}(\mathcal{X})$, then M_z is reflexive.*

Proof. Let $X \in AlgLat(M_z)$. By an argument similar to the proof of Lemma 3.1 in [5] we can show that $X = M_\varphi$ for some multiplier φ. Now we show that $L : H^\infty(\Omega_1) \longrightarrow B(\mathcal{X})$ be given by $L(\varphi) = M_\varphi$ is continuous. Suppose that the sequence $\{\varphi_n\}_n$ converges to φ in $H^\infty(\Omega_1)$ and $L(\varphi_n) = M_{\varphi_n}$ converges to A in $B(\mathcal{X})$. Then for each f in $\mathcal{X}$,

$$Af = \lim_n M_{\varphi_n} f = \lim_n \varphi_n f,$$

and so $\{\varphi_n f\}_n$ is convergent in $\mathcal{X}$. Note that by the continuity of point evaluations $\varphi_n f$ converges pointwise to φf. Thus Af is analytic on Ω

and agree with φf on Ω. Hence $A = M_\varphi$ and so L is continuous. This implies that there is a constant $c_1 > 0$ such that

$$\|M_\varphi\| \leqslant c_1 \|\varphi\|_{\Omega_1},$$

for all φ in $H^\infty(\Omega_1)$.

Now put $\mathcal{N} = H^\infty(\Omega_1)$. Then $\mathcal{N} \neq \emptyset$, since $1 \in \mathcal{N}$. It is a closed subspace of $\mathcal{X}$, since if $\{f_n\}_n \subset \mathcal{N}$ and $f_n \longrightarrow f$ in $\mathcal{X}$, then for all n, $\|f_n\|_{\mathcal{X}} \leqslant c_2$ for some $c_2 > 0$. Because point evaluations are bounded, for all λ in Ω we have

$$f_n(\lambda) =< f_n, e(\lambda) > \longrightarrow < f, e(\lambda) >= f(\lambda).$$

Also for all λ in Ω,

$$|f_n(\lambda)| = | < f_n, e(\lambda) > | \leqslant \|f_n\|_{\mathcal{X}} \|e(\lambda)\| \leqslant c_3 \|f_n\|_{\mathcal{X}},$$

where $c_3 = \sup_{\lambda \in \Omega} \|e(\lambda)\|$. Thus

$$\|f_n\|_\Omega \leqslant c_3 \|f_n\|_{\mathcal{X}} \leqslant c_2 c_3,$$

for all n. Since $f_n \in H^\infty(\Omega_1)$, $\|f_n\|_{\Omega_1} = \|f_n\|_\Omega$ and so $\|f_n\|_{\Omega_1} \leqslant c_2 c_3$ for all n. This implies that $\{f_n\}_n$ is a normal family in $H^\infty(\Omega_1)$ and by passing to a subsequence if necessary, we may suppose that for some function g, $f_n \longrightarrow g$ uniformly on compact subsets of Ω_1. Thus $g \in H^\infty(\Omega_1)$. But by pointwise convergence, $f = g$ on Ω. Then f can be extended to a bounded analytic function on Ω_1, i.e., $f \in H^\infty(\Omega_1)$ and so $\mathcal{N}$ is indeed a closed subspace of $\mathcal{X}$. Now clearly $\mathcal{N} \in Lat(M_z)$, thus $X\mathcal{N} \subset \mathcal{N}$. Since $1 \in \mathcal{N}$ we get $X1 = \varphi \in \mathcal{N} = H^\infty(\Omega_1)$. But Ω_1 is a Caratheodory domain and so by the Farrell-Rubel-Shields Theorem [2, Theorem 5.1, p.151] there is a sequence $\{p_n\}_n$ of polynomials converging to φ such that for all n, $\|p_n\|_{\Omega_1} \leqslant c_4$ for some $c_4 > 0$. So we obtain

$$\|M_{p_n}\| \leqslant c_1 \|p_n\|_{\Omega_1} \leqslant c_1 c_4,$$

for all n. Since $\mathcal{X}$ is reflexive, the unit ball of $\mathcal{X}$ is weakly compact. Therefore ball $B(\mathcal{X})$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(\mathcal{X})$, $M_{p_n} \longrightarrow A$ in the weak operator topology. Using the fact that $M^*_{p_n} \longrightarrow A^*$ in the weak operator topology and by acting these operators on $e(\lambda)$ we obtain that

$$p_n(\lambda)e(\lambda) = M^*_{p_n} e(\lambda) \longrightarrow A^* e(\lambda),$$

weakly. Since $p_n(\lambda) \longrightarrow \varphi(\lambda)$ we see that

$$A^* e(\lambda) = \varphi(\lambda) e(\lambda).$$

Because the closed linear span of $\{e(\lambda) : \lambda \in \Omega\}$ is weak star dense in $\mathcal{X}^*$, we conclude that $A = M_\varphi = X$. This implies that $X \in W(M_z)$ and so M_z is reflexive. This completes the proof. $\square$

References

[1] P. S. Bourdon and J. H. Shapiro, Spectral and common cyclic vectors, *Michigan Math. J.*, 37 (1990), 71-90.

[2] T. Gamelin, *Uniform algebras*, Chelsea, N. Y., 1984.

[3] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, New York, 1971.

[4] D. Sarason, Weak star generators of H^∞, *Pacific J. Math.*, 17 (1966), 519-528.

[5] K. Seddighi and B. Yousefi, On the reflexivity of operators on function spaces, *Proc. Amer. Math. Soc.,* 116 (1992), 45-52.

[6] A. Shields and L. Wallen, The commutants of certain Hilbert space operators, *Ind. Univ. Math. J.,* 20 (1971), 777-788.

[7] B. Yousefi, Interpolating sequences on certain Banach spaces of analytic functions, *Bull. Austral. Math. Soc.,* 65 (2002), 177-182.

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