Reflexivity on Banach Spaces of Analytic Functions

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“Dedicated to Mola Ali”

Abstract. Let $\mathcal{X}$ be a Banach space of functions analytic on a plane domain $\Omega$ such that for every $\lambda$ in $\Omega$ the functional of evaluation at $\lambda$ is bounded. Assume further that $\mathcal{X}$ contains the constants and admits multiplication by the independent variable $z$, $M_z$, as a bounded operator. We give sufficient conditions for $M_z$ to be reflexive.

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1. Introduction

Let $\mathcal{X}$ be a separable reflexive Banach space whose elements are analytic functions on a complex domain $\Omega$. It is convenient and helpful to introduce the notation $\langle x, x^* \rangle$ to stand for $x^*(x)$, for $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. Assume $1 \in \mathcal{X}$ and the operator $M_z$ of multiplication by $z$ maps $\mathcal{X}$ into

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itself and for each \( \lambda \) in \( \Omega \), the functional \( e(\lambda) : \mathcal{X} \to C \) of evaluation at \( \lambda \) given by

\[
e(\lambda)(f) = \langle f, e(\lambda) \rangle = f(\lambda).
\]

is bounded.

For the algebra \( \mathcal{B}(\mathcal{X}) \) of all bounded operators on a Banach space \( \mathcal{X} \), the weak operator topology is the one in which a net \( A_\alpha \) converges to \( A \) if \( A_\alpha x \to Ax \) weakly, \( x \in \mathcal{X} \) ([7]).

A complex valued function \( \varphi \) on \( \Omega \) for which \( \varphi f \in \mathcal{X} \) for every \( f \in \mathcal{X} \) is called a multiplier of \( \mathcal{X} \) and the collection of all these multipliers is denoted by \( \mathcal{M}(\mathcal{X}) \). Because \( M_\varphi \) is a bounded operator on \( \mathcal{X} \), the adjoint \( M_\varphi^* : \mathcal{X}^* \to \mathcal{X}^* \) satisfies

\[
M_\varphi^* e(\lambda) = \lambda e(\lambda).
\]

In general each multiplier \( \varphi \) of \( \mathcal{X} \) determines a multiplication operator \( M_\varphi \) defined by \( M_\varphi f = \varphi f, f \in \mathcal{X} \). Also

\[
M_\varphi^* e(\lambda) = \varphi(\lambda) e(\lambda).
\]

It is well-known that each multiplier is a bounded analytic function. Indeed \( |\varphi(\lambda)| \leq ||M_\varphi|| \) for each \( \lambda \) in \( \Omega \). Also \( M_\varphi 1 = \varphi \in \mathcal{X} \). But \( \mathcal{X} \subset H(\Omega) \), thus \( \varphi \) is a bounded analytic function.

Recall that if \( A \in \mathcal{B}(\mathcal{X}) \), then \( \text{Lat}(A) \) is by definition the lattice of all invariant subspaces of \( A \), and \( \text{AlgLat}(A) \) is the algebra of all operators
$B$ in $\mathcal{B}(X)$ such that $\text{Lat}(A) \subseteq \text{Lat}(B)$. An operator $A$ in $\mathcal{B}(X)$ is said to be reflexive if $\text{AlgLat}(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathcal{B}(X)$ that contains $A$ and the identity $I$ and is closed in the weak operator topology.

For $G$ an open connected (not necessarily simply connected) subset of the complex plane and $\alpha$ an ordinal number, the set $G_\alpha$ is defined as in Sarason [4, p.525]. Here we only remember the definition of the Caratheodory hull. By a domain we understand a connected open subset of the plane. If $B$ is a bounded domain in the plane, then the Caratheodory hull (or $C$-hull) of $B$ is the complement of the closure of the unbounded component of the complement of the closure of $B$. The $C$-hull of $B$ is denoted by $B^*$. Intuitively, $B^*$ can be described as the interior of the outer boundary of $B$, and in analytic terms it can be defined as the interior of the set of all points $z_0$ in the plane such that

$$|p(z_0)| \leq \sup\{|p(z)| : z \in B\},$$

for all polynomials $p$. The components of $B^*$ are simply connected; in fact, one can easily see that each of these components has a connected complement. The component of $B^*$ that contains $B$ is denoted by $B_1$. Note that for all polynomials $p$, $\|p\|_B = \|p\|_{B_1}$. 
2. Main Result

The operator $M_z$ has been the focus of attention for several decades and many of its properties have been studied (e.g. [1],[6]). In this article we would like to give some sufficient conditions so that the operator $M_z$ becomes reflexive (for a good source of reflexivity see [3]). This is a continuation of our work [5] where we only considered finitely connected domains, but here we work with arbitrary domains.

**Theorem.** Let $X$ be a separable reflexive Banach space whose elements are analytic functions on a complex domain $\Omega$, each point of which is a bounded point evaluation. Suppose that $X$ contains the constant functions and $z \in \mathcal{M}(X)$. If $\{e(\lambda) : \lambda \in \Omega\}$ is norm bounded and $H^\infty(\Omega_1) \subset \mathcal{M}(X)$, then $M_z$ is reflexive.

**Proof.** Let $X \in \text{AlgLat}(M_z)$. By an argument similar to the proof of Lemma 3.1 in [5] we can show that $X = M_\varphi$ for some multiplier $\varphi$.

Now we show that $L : H^\infty(\Omega_1) \to B(X)$ be given by $L(\varphi) = M_\varphi$ is continuous. Suppose that the sequence $\{\varphi_n\}_n$ converges to $\varphi$ in $H^\infty(\Omega_1)$ and $L(\varphi_n) = M_{\varphi_n}$ converges to $A$ in $B(X)$. Then for each $f$ in $X$,

$$Af = \lim_n M_{\varphi_n}f = \lim_n \varphi_nf,$$

and so $\{\varphi_nf\}_n$ is convergent in $X$. Note that by the continuity of point evaluations $\varphi_nf$ converges pointwise to $\varphi f$. Thus $Af$ is analytic on $\Omega$. 


and agree with $\varphi f$ on $\Omega$. Hence $A = M_\varphi$ and so $L$ is continuous. This implies that there is a constant $c_1 > 0$ such that

$$\|M_\varphi\| \leq c_1 \|\varphi\|_{\Omega_1},$$

for all $\varphi$ in $H^\infty(\Omega_1)$.

Now put $\mathcal{N} = H^\infty(\Omega_1)$. Then $\mathcal{N} \neq \emptyset$, since $1 \in \mathcal{N}$. It is a closed subspace of $X$, since if $\{f_n\} \subset \mathcal{N}$ and $f_n \to f$ in $X$, then for all $n$, $\|f_n\|_X \leq c_2$ for some $c_2 > 0$. Because point evaluations are bounded, for all $\lambda$ in $\Omega$ we have

$$f_n(\lambda) = \langle f_n, e(\lambda) \rangle \to \langle f, e(\lambda) \rangle = f(\lambda).$$

Also for all $\lambda$ in $\Omega$,

$$|f_n(\lambda)| = |\langle f_n, e(\lambda) \rangle| \leq \|f_n\|_X \|e(\lambda)\| \leq c_3 \|f_n\|_X,$$

where $c_3 = \sup_{\lambda \in \Omega} \|e(\lambda)\|$. Thus

$$\|f_n\|_{\Omega_1} \leq c_3 \|f_n\|_X \leq c_2 c_3,$$

for all $n$. Since $f_n \in H^\infty(\Omega_1)$, $\|f_n\|_{\Omega_1} = \|f_n\|_{\Omega}$ and so $\|f_n\|_{\Omega_1} \leq c_2 c_3$ for all $n$. This implies that $\{f_n\}$ is a normal family in $H^\infty(\Omega_1)$ and by passing to a subsequence if necessary, we may suppose that for some function $g$, $f_n \to g$ uniformly on compact subsets of $\Omega_1$. Thus $g \in H^\infty(\Omega_1)$. But by pointwise convergence, $f = g$ on $\Omega$. Then $f$ can be extended to a bounded analytic function on $\Omega_1$, i.e., $f \in H^\infty(\Omega_1)$ and
so \( \mathcal{N} \) is indeed a closed subspace of \( \mathcal{X} \). Now clearly \( \mathcal{N} \in \text{Lat}(M_x) \), thus \( X\mathcal{N} \subset \mathcal{N} \). Since \( 1 \in \mathcal{N} \) we get \( X1 = \varphi \in \mathcal{N} = H^\infty(\Omega_1) \). But \( \Omega_1 \) is a Caratheodory domain and so by the Farrell-Rubel-Shields Theorem [2, Theorem 5.1, p.151] there is a sequence \( \{p_n\}_n \) of polynomials converging to \( \varphi \) such that for all \( n \), \( \|p_n\|_{\Omega_1} \leq c_4 \) for some \( c_4 > 0 \). So we obtain

\[
\|M_{p_n}\| \leq c_1\|p_n\|_{\Omega_1} \leq c_1c_4,
\]

for all \( n \). Since \( \mathcal{X} \) is reflexive, the unit ball of \( \mathcal{X} \) is weakly compact.

Therefore ball \( B(\mathcal{X}) \) is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some \( A \in B(\mathcal{X}) \), \( M_{p_n} \longrightarrow A \) in the weak operator topology. Using the fact that \( M_{p_n}^* \longrightarrow A^* \) in the weak operator topology and by acting these operators on \( e(\lambda) \) we obtain that

\[
p_n(\lambda)e(\lambda) = M_{p_n}^*e(\lambda) \longrightarrow A^*e(\lambda),
\]

weakly. Since \( p_n(\lambda) \longrightarrow \varphi(\lambda) \) we see that

\[
A^*e(\lambda) = \varphi(\lambda)e(\lambda).
\]

Because the closed linear span of \( \{e(\lambda) : \lambda \in \Omega\} \) is weak star dense in \( \mathcal{X}^* \), we conclude that \( A = M_\varphi = X \). This implies that \( X \in W(M_z) \) and so \( M_z \) is reflexive. This completes the proof. \( \square \)
References


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