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P-Dense Submodules

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Abstract. Let M and P be right $R-$modules. A submodule K of an $R-$module M is called $P-$dense if for each $m \in M, (K : m)$ is a $P-$faithful right ideal of R. P_R is nonsingular if and only if, for each $R-$module M, every essential submodule of M is a $P-$dense submodule. For any $R-$module M, we obtain $P-$rational extention of M and equivalent condition in order that M is equal with its $P-$rational extention is found. An $R-$module P is called right Kasch if every simple $R-$module can be embedded in P. Finally, we given some equivalent conditions for an $R-$module P to be right Kasch.

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1. Introduction

Throughout this paper, all rings are associative with non-zero identity and modules are unitary right module. For any $R-$module M, $E(M)$ is injective hull of M. Let $\mathcal{A}$ be a set of right ideals of R and K be a submodule of an $R-$module M. K is called $\mathcal{A}$-submodule of M, if for each $m \in M$, $(K : m) \in \mathcal{A}$. K denoted by $K \subseteq_{\mathcal{A}} M$. Let P be a right

R-module and I be a right ideal of R. I is called P-faithful, if

$$ann_P(I) = \{p \in P : pI = 0\} = 0.$$

Let $\mathcal{A}$ be the set of all P-faithful right ideals of R and K be a submodule of M. If $K \subseteq_{\mathcal{A}} M$, then K is called a $P-$dense submodule. Dense submodules have been investigated by several authors some of their recent work are cited in the reference. If K is a P-dense submodule of M, then we denote it by $K \subseteq_P M$. In Section 2, we study properties of P-dense submodules of an R-module M. Equivalent conditions are given for a submodule to be P-dense are found 2.4. and show that P is a nonsingular R-module if and only if for each right R-module M, its essential submodules are P-dense submodules. In Section 3, we study modules are called right Kasch R-modules.

2. P-Dense Submodules

Lemma 2.1. *Let M and P be right R-modules and K be a submodule of M. The following are equivalent.*
(i) K is a P-dense submodule of M.
(ii) For each $m \in M$ and non-zero element $p \in P$, there exists $r \in R$ such that $pr \neq 0$ and $mr \in K$.

proof. $(i) \Rightarrow (ii)$ Let $m \in M$ and $0 \neq p \in P$. By (i), $(K : m)$ is P-faithful, then $p(K : m) \neq 0$.

$(ii) \Rightarrow (i)$ Let $m \in M$ and $p \in P$ such that $p(K : m) = 0$. If $p \neq 0$, by (ii), there exists $r \in R$ such that $pr \neq 0$ and $mr \in K$. Thus $p(K : m) \neq 0$, a contradiction. □

Example 2.2. If K is a dense submodule of a right module M over a ring R, then K is M-dense.

Example 2.3. Every essential submodule of a Z-module is Z-dense.

Theorem 2.4. *Let M and P be R-modules and K be a submodule of M. The following are equivalent.*

(i) $K \subseteq_P M$.

(ii) $Hom_R(M/K, E(P)) = 0$.

(iii) For any submodule Q of M such that $K \subseteq Q \subseteq M, Hom_R$ $(Q/K, P) = 0$.

Proof. $(i) \Rightarrow (ii)$ Let $f \in Hom_R(M, E(P))$ be such that $f(K) = 0$. If $f \neq 0$, then $f(m) \neq 0$ for some non-zero element $m \in M$. Since $P \subseteq_{ess} E(P)$, there exists $r \in R$ such that $0 \neq f(mr) = f(m)r \in P$. Since $K \subseteq_P M$, there exists $s \in R$ such that $f(mr)s = f(mrs) \neq 0$ and $mrs \in K$, a contradiction.

$(ii) \Rightarrow (iii)$ Assume that, for some Q as in (iii), there exists a non-zero R-homomorphism $g \in Hom_R(Q, P)$ such that $g(K) = 0$. Since $E(P)$ is an injective R-module, then g can be extended to $\bar{g} \in Hom_R(M, E(P))$.

Since $\bar{g}(K) = g(K) = 0$, by (ii), $\bar{g} = 0$, a contradiction.

$(iii) \Rightarrow (i)$ Suppose that $p(K : m) = 0$ fore some $m \in M$ and $p \in P \backslash \{0\}$. We define $f : K + mR \to P$ by

$$f(k + mr) = pr \qquad (k \in K, r \in R).$$

This map is well-defined, for, if $k + mr = \acute{k} + m\acute{r}$, then $(k - \acute{k}) = m(\acute{r} - r) \in K$. Hence $p(\acute{r} - r) = 0$. Clearly, f is an R-homomorphism vanishing on K. So by (iii), $0 = f(m) = p$, a contradiction. $\square$

Proposition 2.5. *Let M and P be R-modules and K and L be submodules of M. Then*

(i) If $K \subseteq_P M$, $L \subseteq_P M$, then $K \cap L \subseteq_P M$.

(ii) Let $L \subseteq K \subseteq M$. Then $L \subseteq_P M$ if and only if $L \subseteq_P K$ and $K \subseteq_P M$.

Proof. (i) Let $m \in M$ and $p \in P \setminus \{0\}$. Since $K \subseteq_P M$, there exists $r \in R$ such that $pr \neq 0$ and $mr \in K$. Since $L \subseteq_P M$, there exists $s \in R$ such that $prs \neq 0$ and $mrs \in L$. Thus $prs \neq 0$ and $mrs \in K \cap L$.

(ii) It is sufficient to prove the "if " part. Assume that $L \subseteq_P K$ and $K \subseteq_P M$. Let $m \in M$ and $p \in P \setminus \{0\}$. There exists $r \in R$ such that $pr \neq 0$ and $mr \in K$ since $K \subseteq_P M$. Since $L \subseteq_P K$, there exists $s \in R$ such that $prs \neq 0$ and $mrs \in L$. $\square$

Let M be a right module over a ring R. An element $m \in M$ is said to be singular element of M if the right ideal $ann_r(m)$ is essential in R_R. The set of all singular elements of M is denoted by $Z(M)$. The right $R-$module M is called nonsingular module, if $Z(M) = 0$.

Theorem 2.6. *For any R-module P, the following are equivalent.*

(i) *P is a nonsingular R-module.*

(ii) *Every essential submodule of any R-module M is a P-dense submodule.*

Proof. $(i) \Rightarrow (ii)$ Let M be a right $R-$module and $K \subseteq_{ess} M$. Let $m \in M$ and $p \in P \setminus \{0\}$. Since K is an essential submodule of M, then $(K : m) \subseteq_{ess} R$. If $p(K : m) = 0$, then $(K : m) \subseteq ann_r(p) \subseteq R$. Since $(K : m) \subseteq_{ess} R$, then $ann_r(p) \subseteq_{ess} R$ and hence $p \in Z(P)$, a contradiction.

$(ii) \Rightarrow (i)$ Let $p \in Z(P)$. Then $ann_r(p) \subseteq_{ess} R$. By (ii), $ann_r(p) \subseteq_P R$. If $p \neq 0$, then there exists $r \in R$ such that $pr \neq 0$ and $1.r \in ann_r(p)$, a contradiction. $\square$

Definition 2.7. *For two R-modules M and P, we define*

$$\tilde{E}_P(M) = \{x \in E(M) : \forall f \in Hom_R(E(M), E(P)), f(M) = 0 \Rightarrow f(x) = 0\}.$$

Lemma 2.8. *Let P and M be R−modules and N be any submodule of E(M) containing M. Then* $M \subseteq_P N$ *if and only if* $N \subseteq \tilde{E}_P(M)$.

Proof. For the "if " part, it suffices to show that $Hom_R(N/M, E(P)) =$

0. Assume that, $f \in Hom_R(N, E(P))$ such that $f(M) = 0$. Since $E(P)$ is an injective R-module, then f can be extended to $\bar{f} \in Hom_R(E(M), E(P))$. Since $\bar{f}(M) = f(M) = 0$ and $N \subseteq \tilde{E}_P(M)$, then $f(N) = \bar{f}(N) = 0$. Hence $f = 0$. For the "only if" part, assume that $M \subseteq_P N$ and consider $f \in Hom_R(E(M), E(P))$ such that $f(M) = 0$. If $f(N) \neq 0$, then there exists $n \in N \setminus \{0\}$ such that $f(n) \in E(P) \setminus \{0\}$. Since $P \subseteq_{ess} E(P)$, there exists $r \in R$ such that $f(n)r = f(nr) \in P \setminus \{0\}$. For $nr \in N$ and $f(nr) \in P \setminus \{0\}$, $M \subseteq_P N$ implies that $f(nr).s = f(nrs) \in P \setminus \{0\}$ and $nrs \in M$, for some $s \in R$. It is a contradiction, since $f(M) = 0$. □

Proposition 2.9. *For two R-modules M and P, we have*

$$\tilde{E}_P(M) = \{m \in E(M) : \forall x \in E(P) \setminus \{0\}, x(M : m) \neq 0\}.$$

Proof. Let $m \in \tilde{E}_P(M)$ and $x \in E(P) \setminus \{0\}$. Since P is an essential submodule of $E(P)$, there exists $r \in R$ such that $xr \in P \setminus \{0\}$. By Lemma 2.8. $M \subseteq_P \tilde{E}_P(M)$ and hence there exists $s \in R$ such that $xrs \neq 0$ and $mrs \in M$, hence $x(M : m) \neq 0$. Conversely, assume $m \in RHS$ and $f \in Hom_R(E(M), E(P)) \neq 0$ such that $f(M) = 0$. If $f(m) \neq 0$, then by hypothesis, $f(m)(M : m) \neq 0$. Thus there exists $r \in R$ such that $f(m)r = f(mr) \neq 0$ and $mr \in M$. It is a contradiction, since $f(M) = 0$. □

Definition 2.10. *An R-module M_R is called rationally* P- *complete if*

$\tilde{E}_P(M) = M$.

Theorem 2.11. *For any R-modules M and P, the following are equivalent.*

(i) M is rationally P-complete.

(ii) For any R-modules $A \subseteq B$ such that $Hom_R(B/A, E(P)) = 0$, any R-homomorphism $f : A \to M$ can be extended to B.

Proof. $(i) \Rightarrow (ii)$. Let $A \subseteq B$ be R-modules such that $Hom_R(B/A, E(P)) = 0$, and let $f \in om_R(A, M)$. Since $E(M)$ is an injective R-module, We can extended f to $g : B \to E(M)$. We claim that

$$M \subseteq_P M + g(B).$$

Once we have proved this, then lemma 2.8. and (i) imply that $g(B) \subseteq M$ and we are done. By theorem 2.4. it suffices to prove that

$$Hom_R((M + g(B))/M, E(P)) = 0.$$

Suppose $h \in Hom_R((M + g(B))/M, E(P))$. Define R-homomorphism $\acute{h} \in Hom_R(B/A, (M+g(B))/M)$ by $\acute{h}(b+A) = g(b)+M$ $(b \in B)$. Since $Hom_R(B/A$,
E$(P)) = 0$, $h\acute{h} = 0$. Hence for each $m \in M$ and $b \in B$,

$$h((m + g(b)) + M) = h(g(b) + M) = ho\acute{h}(b + A) = 0.$$

Therefore $h = 0$. $(ii) \Rightarrow (i)$ By theorem 2.4. $M \subseteq_P \tilde{E}_P(M)$, and so $Hom_R(\tilde{E}_P$

$(M)/M, E(P)) = 0$. By (ii), the identity map $id : M \to M$ can be extended to an $R-$module $g : \tilde{E}_P(M) \to M$ such that $gi = id$, where $i : M \to \tilde{E}_P(M)$ is an inclusion map. Therefore $\tilde{E}_P(M) = \ker g \oplus Im\, i$. Since $\ker g \cap M = 0$ and M is an essential submodule of $E(M)$, then $\ker g = 0$. Hence

$$\tilde{E}_P(M) = Im\ \ i = M. \ \square$$

3. Right Kasch R-modules

If S is a simple right module and P is a right module over a ring R, it is of interest to know whether S can be embedded in P. Consideration of this issue leads to the notion of right Kasch R-modules.

Definition 3.1. *Let P be a right module over a ring R. We say that P is right Kasch $R-$module if every simple right $R-$module S can be embedded in P.*

Theorem 3.2. *Let P be a right module over a ring R. For any maximal right ideal m of R, the following are equivalent.*

(i) R/m embeds to P_R.

(ii) $m = ann_r(x)$ for some $x \in P$.

(iii) m is not a $P-$faithful right ideal.

(iv) $m = ann_r(ann_P(m))$.

(v) m is not $P-$dense in R_R.

Proof. $(i) \Rightarrow (ii)$ Let $f \in Hom_R(R/m, P)$ be a monomorphism and $0 \neq x = f(1+m)$. For each $r \in m$,

$$xr = f(1+m)r = f(r+m) = f(m) = 0.$$

Therefore $r \in ann_r(x)$. Hence $m \subseteq ann_r(x)$. Since $ann_r(x) \neq R$, then $m = ann_r(x)$.

$(ii) \Rightarrow (iii)$ Let $m = ann_r(x)$ for some non-zero element $x \in P_R$. Then $xm = 0$, therefore m is not a P-faithful right ideal of R.

$(iii) \Rightarrow (iv)$ We know that $ann_P(m).m = 0$, then $m \subseteq ann_r(ann_P(m))$. If $ann_r(ann_P(m)) = R$, then $ann_P(m) = 0$. Hence m is a P-faithful, a contradiction. Therefore $ann_r(ann_p(m))$ is a proper right ideal of R and hence

$$m = ann_r(ann_P(m)).$$

$(iv) \Rightarrow (v)$ Since $ann_P(m) \neq 0$, there exists $0 \neq p \in ann_P(m)$.

For $1 \in R$, $(m : 1) = m$, and hence

$$pm = p(m : 1) = 0.$$

Therefore m is not P-dense in R_R, by Lemma 2.1.

$(v) \Rightarrow (i)$ By Theorem 2.4. since m is not P-dense in R_R, then Hom_R $(R/m, E(P)) \neq 0$. Let f be a non-zero element of $Hom_R(R/m, E(P))$. Since R/m is a simple R-module, f is a monomorphism and hence $Im\ f \simeq R/m$. Since P is an essential submodule of $E(P)$, then $Im\ f \cap$

$P \neq 0$. Since $Im\ f$ is a simple R-module, then $Im\ f \cap P = Im\ f$. Therefore $Im\ f \subseteq P$. This show that $f : R/m \to P$ is a monomorphism. $\square$

Corollary 3.3. *For any right R-module P, the following are equivalent.*

(i) P is a right Kasch module.

(ii) Any maximal right ideal in R has the form $ann_r(x)$ for some $x \in P$.

(iii) The set of P-faithful right ideals of R does not contain any maximal right ideal of R.

(iv) For any maximal right ideal m of R, $ann_r(ann_P(m)) = m$.

(v) The only P-dense right ideal in R is R itself.

(vi) For any proper right ideal I of R, $ann_P(I) \neq 0$.

Proof. $(i) \Rightarrow (ii)$ Let m be a maximal right ideal of R, then the simple R-module R/m embeds in P. By Theorem 3.1, $m = ann_r(x)$ for some $x \in P$.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ By Theorem 3.1.

$(iv) \Rightarrow (v)$ Let J be a proper P-dense right ideal of R_R. There exists a maximal right ideal m containing J. By Proposition 2.5. m is a P-dense in R_R. By Theorem 3.1. $m \neq ann_r(ann_P(m))$, a contradiction.

$(v) \Rightarrow (i)$ For each maximal right ideal m of R, m is not P-dense in R. By Theorem 3.2. R/m embeds in P. Which implies that P is a right Kasch R-module.

$(ii) \Rightarrow (vi)$ Let I be a proper right ideal of R. There exists a maximal right ideal m contains I. By (ii), $m = ann_r(x)$ for some $0 \neq x \in P$. Then $I \subseteq m = ann_r(x)$, and so $xI = 0$. It implies that $ann_P(I) \neq 0$.

$(vi) \Rightarrow (iii)$ For any maximal right ideal m of R, $ann_P(m) \neq 0$. Then m is not P-faithful. $\square$

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