Journal of Mathematical Extension
Vol. 3, No. 2 (2009), 43-59

Goodness of Fit Test and Test of Independence by Entropy

M. Sharifdoost
Islamic Azad University Science & Research Branch, Tehran

N. Nematollahi
Allameh Tabatabaie University

E. Pasha
Teacher Training University

Abstract. To test whether a set of data has a specific distribution or not, we can use the goodness of fit test. This test can be done by one of Pearson X^2-statistic or the likelihood ratio statistic G^2, which are asymptotically equal, and also by using the Kolmogorov-Smirnov statistic in continuous distributions. In this paper, we introduce a new test statistic for goodness of fit test which is based on entropy distance, and which can be applied for large sample sizes. We compare this new statistic with the classical test statistics X^2, G^2, and T_n by some simulation studies. We conclude that the new statistic is more sensitive than the usual statistics to the rejection of distributions which are almost closed to the desired distribution. Also for testing independence, a new test statistic based on mutual information is introduced.

AMS Subject Classification: 62G99; 62H10.
Keywords and Phrases: Chi-squared test, goodness of fit test, test of independence, Kolmogorov-Smirnov test, likelihood ratio test, mutual information, relative entropy.

1. Introduction

One of the interesting problems in statistics is finding a distribution which fits to a given set of data. In other words, we want to test whether a specific distribution coincides with given data or not. In a goodness of fit test, we compare an unknown distribution q of a random variable, or a random sample, with a given known distribution p.

There are many ways to test goodness of fit. Karl Pearson [18] introduced a statistic for goodness of fit test that is asymptotically equal to the likelihood ratio statistic for large sample sizes, and it has the Chi-squared distribution. Moreover, for testing independence of two random variables, we can use this statistic and compare it to quantiles of Chi-squared distribution.

Sometimes we can compare two distributions by measuring the distance between them with a suitable criterion. One such comparison is Kolmogorov-Smirnov distance $d(F,G) = \sup_t |F(t) - G(t)|$ for continuous distributions F and G. The Kolmogorov-Smirnov distance is zero if and only if F=G (Lehmann and Romano, [14], p. 584). Another way of measuring the distance between two distributions is relative entropy. Also this measure is zero if and only if the two specified distributions are equal (Cover and Thomas, [5], p. 26). By using the concept of relative entropy, we want to construct a test statistic for testing goodness of fit.

The concept of entropy was first introduced in thermodynamics, where it was used to provide a statement of the second law of thermodynamics. Later, statistical mechanics provided a connection between macroscopic property of entropy and microscopic state of the system. This work was the crowning achievement of Boltzman [4]. In ([12]) Hartley introduced a logarithmic measure of the alphabet size. Shanon [20] was the first who defined entropy and mutual information as defined in this paper.

Entropy of a random variable is the measure of uncertainty of that random variable, i.e., measure of the amount of information required on the average to describe random variable. Entropy has many applications in statistical science and engineering. One of the subjects in information theory is mutual information, i.e., the amount of information that one random variable has from other random variables. On the other hand, it is the uncertainty of one random variable with knowledge of other random variable. If the mutual information is zero, then the two random variables are independent. Taking this into consideration, we introduce a measure based on mutual information for testing independence of two random variables in contingency tables.

There are ideas about the construction of the test of goodness of fit based on entropy but they are based on maximum entropy principal; considering a class of densities satisfying criterion restriction and finding a member of this class that maximizes entropy and finding its parametric consistent estimators. Based on this principal, one can find the test statistic of some densities that maximizes the entropy of special class, including uniform, normal, exponential and inverse Gaussian.

Vasicek [21] proposed an entropy-based test for composite hypothesis of normality and provided the critical values and power with Monte Carlo simulation. Dudewicz and Meulen [7,8,9] extended Vasicek's work and proposed an entropy-based test for uniformity and provided the critical values and power by Monte Carlo simulation. Gokhale [10] proposed the entropy-based test construction for a broad class of distributions. Mudholkar and Lin [16] introduced an entropy-based test for exponential hypothesis and prepared the critical values and power by Monte Carlo simulation. Parzen [17] introduced entropy-based test statistic based on difference of order statistic to test goodness of fit of parametric model $\{\mathrm{F}(x, \theta)\}$. Mergel [15] found a test statistic based on maximum entropy for null hypothesis inverse Gaussian with different estimator.

For testing independence, Robinson [19] introduced a test based on an estimator of Kullback-Leibler divergence and studied consistency on testing serial independence for time series. Zheng [22] claimed: " Robinson's test does not have good power against a broad range of alternatives. Morever, the regularity assumptions imposed by the test are so strong that is rules out even some commonly used distribution such as normal."

Goria et al. [11] constructed goodness of fit tests for normal, Laplace, exponential, Gamma, Beta based on maximum entropy principal and introduced a consistent test of independent.

Our test statistics is based on property of relative entropy that is zero if and only if two distributions are equal. This test can be used for every distribution in null and alternative hypotheses.

In Section 2 we review the definition of relative entropy and mutual information and their properties. In Section 3, we review the goodness of fit test and three well-known statistics. In Section 4, we introduce the new statistic for testing goodness of fit based on relative entropy and derive its asymptotic distribution by using limit theorems. To compare the new test procedure with the usual goodness of fit tests, we provide some examples in Section 5. In Section 6, it is shown by some simulation studies that in contrast to the usual tests, the new test is sensitive to the rejection of distributions which are almost close to the desired distribution. In Section 7, a new test statistic based on mutual information for testing independent is derived, and some examples are given. Finally, in Section 8, some conclusions are given.

2. Elementary Concepts

Relative entropy was first defined by Kullback and Leibler (1951). It is known under a variety of names including the Kullback-Leibler distance, cross entropy, information diverges and information for discrimination, and has been studied in detail by Csiszar (1967) and Amari (1985).
In this section, we review two related concepts; namely, relative entropy and mutual information.

2.1. Relative Entropy

The relative entropy is a measure of the distance between two distributions. It arises on an expected logarithm of the likelihood ratio. The relative entropy $D\left(q\,\|p\right)$ is a measure of the inefficiency of assuming that the distribution is p when the true distribution is q.

Definition 1. *The relative entropy or Kullback-Leibler distance between two probability mass functions $p(x)$ and $q(x)$ is defined as:*

$$D\left(q\,\|p\right) = \sum_{x\in S(x)} q\left(x\right)\log_2\frac{q\left(x\right)}{p\left(x\right)} = \mathrm{E}_q\left[\log_2\frac{q\left(X\right)}{p\left(X\right)}\right], \tag{1}$$

where $S(x)$ is the support of random variable X.
In the above definition, we use the convention (based on continuity argument) that $0\log\frac{0}{p}=0$ and $q\log\frac{q}{0}=\infty$.

The relative entropy is always non-negative and is zero if and only if $p = q$. However, it is not a true distance between two distributions because it is not symmetric and does not satisfy the triangle inequality (Cover and Thomas, 1991, p. 26).

2.2. Mutual Information

The mutual information is a measure of the amount of information that one random variable contains about another random variable. It is the reduction in the uncertainty of one random variable due to knowledge of the other.

Definition 2. *Let X and Y be two random variables with joint probability mass functions $p(x,y)$ and marginal probability mass functions $p(x)$ and $p(y)$. The mutual information $I(X,Y)$ is the relative entropy between the joint distribution and the product $p(x)p(y)$, i.e.,*

$$\begin{aligned} I(X,Y) &= \sum_x \sum_y p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)} \\ &= D(p(x,y)||p(x)p(y)) \\ &= E_{p(X,Y)} \left[\log_2 \frac{p(X,Y)}{p(X)p(Y)} \right] \end{aligned} \tag{2}$$

In discrete case, the property of the mutual information is the same as that of relative entropy, i.e., it is always non-negative. This criterion is the measure of dependence of the two random variables and is zero if and only if X and Y are independent. It is symmetric with respect to X and Y.

3. Goodness of Fit Test

Suppose the result of a random experiment belongs to one of k disjoint mutual categories $A_1, A_2, \ldots, A_k$, where the probability of belonging to category A_i is q_i, $(0 \leqslant q_i \leqslant 1)$, $\sum_{i=1}^{k} q_i = 1$, this unknown probability distribution is denoted by $q(x)$. We want to test whether this random experiment has a known probability distribution $p(x)$ that is $p\left(A_i\right) =$

p_i , $(0 \leqslant p_i \leqslant 1)$, $\sum\limits_{i=1}^{k} p_i = 1$. In other words, we want to test $H_0 : q(x) = p(x)$ for all x versus $H_1 : q(x) \neq p(x)$ for at least one x. To do this test, we repeat the random experiment n times and let O_i be the frequency of results that belong to A_i. If H_0 is true, then the expected frequency of results that belong to category A_i (O_i) is np_i, that is, $e_i = E(O_i) = np_i$ in n experiments.

We can test the above hypothesis by the following well-known test statistic:

1. Pearson statistic $X^2 = \sum\limits_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$. For large sample sizes, X^2 has approximately Chi-squared distribution with $k-1$ degrees of freedom. Therefore, we reject H_0 if $X^2 > \chi^2(k-1, \alpha)$ where $\chi^2(k-1, \alpha)$ is the $1-\alpha$ quantile of Chi-squared distribution. This test is asymptotic maximin at level of α (Lehmann and Romano, 2005, p.593).

2. Likelihood ratio statistic $G^2 = 2\sum\limits_{i=1}^{k} O_i \log_2\left(\frac{O_i}{e_i}\right)$. For large sample sizes, G^2 has Chi-squared distribution with k-1 degrees of freedom. Therefore, reject H_0 if $G^2 > \chi^2(k-1, \alpha)$. The statistic G^2 is asymptotically equivalent to Pearson statistic X^2 (Cover and Thomas, [5], p. 333).

3. The Kolmogrov-Smirnov statistic $T_n = \sup\limits_{x \in S(x)} n^{\frac{1}{2}} |F_n(x) - F(x)|$ where $F_n(x)$ is the empirical distribution of sample and F is the distribution function of continuous random variable X. Reject H_0 if T_n is more than critical value in related tables (Lehmann and Romano, [14], p.584).

In the next section, we introduce another statistic based on relative entropy for testing goodness of fit.

4. The Goodness of Fit Test Based on Relative Entropy

Consider testing the hypothesis $H_0 : q = p$ versus $H_1 : q \neq p$. Using the relative entropy given in definition 1. The above testing problem is equivalent to testing the hypothesis $H_0 : D(q \| p) = 0$ versus $H_1 :$

$D(q\|p) > 0$. For testing these hypotheses, we know that $D(q\|p) = \sum\limits_{i=1}^{k} q_i \log_2 \frac{q_i}{p_i} = E_q\left[\log_2 \frac{q}{p}\right]$.

Let O_i and e_i , $i = 1, \ldots, k$, be the values that are given in Section 3, then the maximum likelihood (ML) estimator of q_i is $\hat{q}_i = \frac{O_i}{n}$. Since $p_i = \frac{e_i}{n}$, so the ML estimator of $D(q\|p)$ is given by:

$$\hat{D}\,(\,q\,\|\,p\,) = \frac{1}{n}\sum_{i=1}^{k} O_i \log_2 \frac{O_i}{e_i}. \tag{1}$$

We know that $O = (O_1, O_2, \ldots, O_k)$ has multinomial distribution, i.e.,

$$O = (O_1, O_2, \ldots, O_k) \sim M_k(n, (q_1, q_2, \ldots, q_k)).$$

For a large sample size, O has an asymptotic multivariate normal distribution $N_k\,(nq, n\,(D_q - qq'))$, where D_q is a diagonal matrix with diagonal elements q_i, $i = 1, 2, \ldots, k$, and $q = (q_1, q_2, \ldots, q_k)$. Therefore,

$$\sqrt{n}\left(\frac{1}{n}O - q\right) \xrightarrow{d} N_k\left(0, D_q - qq'\right),$$

where $\xrightarrow{d}$ denotes the convergence in distribution (Agresti, [1], p. 580). With simple algebra and using limit theorems and equations (1) and (3), we can show:

$$Z = \sqrt{n}\left(\frac{\hat{D}\,(q\|p) - D\,(q\|p)}{\hat{\sigma}}\right) \xrightarrow{d} N\,(0, 1)\,, \tag{2}$$

where

$$\hat{\sigma}^2 = \frac{1}{n}\left[\sum_i O_i \left(\log_2 \frac{O_i}{e_i}\right)^2 - \left(\sum_i O_i \log_2 \frac{O_i}{e_i}\right)^2\right]. \tag{3}$$

So, from the above argument and asymptotic distribution of Z given by (4), in testing the hypothesis $H_0 : D(q\|p) = 0$ versus $H_1 : D(q\|p) > 0$, we can reject H_0 if $Z_0 > Z_\alpha$ where

$$Z_0 = \frac{\sqrt{n}\hat{D}\,(q\|p)}{\hat{\sigma}}, \tag{4}$$

and Z_α is the 1- α quantile of standard normal distribution.

Remark: The above statistic is very close to G^2, but we will see by simulation study, it is more sensitive than G^2.
Note that we cannot compare the power of the new test procedure to the usual goodness of fit tests based on X^2, G^2 and T_n statistics, since the class of alternatives typically is enormously large and can no longer be described by a parametric model (Lehmann and Romano, 2005, p.583). Instead, in Section 6 we use some simulation studies to compare the p-values of these test procedures and compare the sensitivity of these test statistics. Before doing this, we will take a look at some examples in the next section.

5. Examples

In this section, some examples for using and comparing the test procedures that considered in previous sections are given (Bhattacharyya and Johnson,1977).

Example 1. From a large population, a sample of 200 is selected and the number of times that they go to an insurance company in a period of 4-years is recorded in the following table. We want to test that the distribution of data follows a Poisson distribution.

Number	0	1	2	3	4	5	6	7	sum
Frequency	22	53	58	39	20	5	2	1	200

Let X be the number of times that a person goes to the insurance company. We want to test $H_0 : X \sim P(\lambda)$ versus $H_1 : X \nsim P(\lambda)$. Using test statistic (6), we have $Z_0 = 0.8012$ and $Z_{0.05} = 1.645$. So, H_0 is accepted. The Pearson statistic is $X^2 = 2.33$, which is not greater than $\chi^2(4, 0.05) = 9.487$. So, again the hypothesis H_0 is accepted.

Example 2. In a general election in a country about a subject, the percentage of answers is:

opinion	very agreeable	agreeable	abstention	disagreeable	very disagreeable
percentage	20	30	20	20	10

A sample of 100 is selected and the results are collected in the following table.

opinion	very agreeable	agreeable	abstention	disagreeable	very disagreeable
frequency	14	18	18	26	24

Do these data coincide with general election?
By considering the above tables and using (6),we have $Z_0 = 2.3156$, and by comparing it to $Z_{0.05} = 1.645$, we conclude that the hypothesis is rejected. Also $X^2 = 46.5$ and $\chi^2(4, 0.05) = 9.487$, so at the level $\alpha = 0.05$, we conclude that H_0 is rejected and the two methods have the same conclusion.

Example 3. The following table shows the results of tossing a dice 120 times. We want to test whether the dice is biased or not.

spot	1	2	3	4	5	6	sum
frequency	18	23	16	21	18	24	120

By considering $p_i = \frac{1}{6}$ and $e_i = 20$, we have $Z_0 = 0.7914$. Comparing it with $Z_{0.05} = 1.645$, we conclude that H_0 is accepted. Also, $X^2 = 2.5$ and comparing it with $\chi^2(5, 0.05) = 11.1$, we conclude that H_0 is also not rejected.

6. Simulation Study

As we told in Section 4, the class of alternative hypothesis is very large and contains all distributions except the distribution given in H_0. Thus, we cannot compute and compare the power function of the new test procedure with the usual ones. In this section, we generate random numbers from Poisson and exponential distributions and carry out the goodness of fit test by the test statistics which are discussed in this paper. To compare these test functions, we use their p-values.

6.1. Poisson Test

In this section, we simulate 10000 random numbers from $P(\lambda_1)$ for $\lambda_1 = 1, 2$ and consider testing hypothesis $H_0 : X \sim P(\lambda_2)$ versus $H_1 : X \nsim P(\lambda_2)$, where values of λ_2 are given in Tables 1 and 2, and compute values of $\hat{d} = \hat{D}(q \| p)$, $\hat{\sigma}^2 = S_d^2$, Z, X^2, G^2 and p-values corresponding to Z, X^2 and G^2 statistics. Since the data are generated from Poisson distribution with $\lambda_1 = 1$, from Table 1, we see that for values of λ_2 near 1(rows I), all procedures accept H_0 and for values of λ_2 far from 1(rows III), all procedures reject H_0. But, for values of λ_2 between (1.0196 , 1.02) the new method based on Z statistic reject H_0 (which is correct) but the usual methods based on X^2 and G^2 accept H_0 (which is not correct). This shows that the new statistic Z is more sensitive than the usual statistic to the rejection of distributions which are almost close to the desired distribution.

Table 1. Results of testing $H_0 : X \sim P(\lambda_2)$ versus $H_1 : X \nsim P(\lambda_2)$ for 10000 random numbers generated from P(1).

	λ_2	$\hat{d}$	S_d^2	Z	X^2	G^2	p-value (z)	p-value (X^2)	p-value (G^2)
I	1	0.0008	0.0048	1.15	2.32	2.36	0.1251	0.89	0.88
	1.005	0.0008	0.0049	1.24	3.21	3.29	0.1075	0.7788	0.774
	1.01	0.0009	0.005	1.35	4.58	4.71	0.0855	0.6	0.584
	1.015	0.0011	0.005	1.49	6.43	6.61	0.068	0.4	0.37
	1.019	0.0012	0.006	1.63	8.25	8.48	0.0526	0.22	0.215
	1.0195	0.0013	0.006	1.646	8.49	8.74	0.0505	0.21	0.201
II	1.0196	.0013	0.006	1.65	8.54	8.79	0.0495	0.21	0.198
	1.0197	0.0013	0.006	1.654	8.59	8.84	0.0495	0.209	0.196
	1.02	.0013	0.006	1.66	8.75	8.99	0.0485	0.2	0.188
III	1.03	0.0017	0.007	2.05	14.77	15.2	0.0202	0.023	0.02
	1.04	0.0023	0.0085	2.5	22.6	23.27	0.0062	0	0
	1.05	0.003	0.001	2.96	32.18	33.18	0.0015	0	0

Table 2. Results of testing $H_0 : X \sim P(\lambda_2)$ versus $H_1 : X \nsim P(\lambda_2)$ for 10000 random numbers generated from $P(2)$.

	λ_2	$\hat{d}$	S_d^2	Z	X^2	G^2	p-value (z)	p-value (X^2)	p-value (G^2)
I	2	0.001	0.005	1.58	3.72	3.81	0.0571	0.88	0.86
	2.001	0.001	0.005	1.6	3.87	3.96	0.0548	0.86	0.855
	2.005	0.0017	0.005	1.65	4.53	4.64	0.0495	0.8	0.79
II	2.01	0.0012	0.005	1.73	5.58	5.72	0.0418	0.69	0.68
	2.03	0.002	0.006	2.17	12.17	12.47	0.0150	0.16	0.14
	2.04	0.002	0.008	2.45	16.89	17.3	0.0071	0.033	0.028
III	2.05	0.003	0.008	2.75	22.52	23.08	0.003	0	0
	2.06	0.003	0.0095	3.06	29.07	29.82	0.0011	0	0
	2.07	0.0035	0.011	3.38	36.52	37.48	0	0	0

Similar results can be seen from Table 2, where we generate 10000 random numbers from P(2) and then test the hypothesis $H_0 : X \sim P(\lambda_2)$ versus $H_1 : X \nsim P(\lambda_2)$ for some values of λ_2. Again we see that for values $\lambda_2 \in (2.005, 2.03)$ the new method based on Z statistic rejects H_0 but the other methods accept it.

6.2. Exponential Test

Similar to the previous section, we simulate 10000 random numbers from exp(1) and consider testing hypothesis $H_0 : X \sim exp(\lambda_2)$ versus $H_1 : X \nsim exp(\lambda_2)$, where values of λ_2 are given in Table 3 and computed $\hat{d}$, $\hat{\sigma}^2$, Z, X^2, G^2, T_n and p-values corresponding to Z, X^2, and G^2 statistics where the critical value of T_n is 1.36. We see that for values of λ_2 near 1 (rows IV), all procedures accept H_0 and for $\lambda_2 = 1.06$ that is far from 1 (row I), all procedures reject H_0. For values of λ_2 in (1.039,1.041) the new method based on Z statistic rejects H_0 but the usual methods based on X^2, G^2, and T_n accept H_0. This shows the new method is more sensitive than the others to the rejection of distribution which are almost close to the desired distribution. Although we see in row III for $\lambda_2 = 1.05$, the methods based on X^2 and G^2.

Table 3. Results of testing $H_0 : X \sim \exp(\lambda_2)$ versus $H_1 : X \nsim \exp(\lambda_2)$ for 10000 random numbers generated from exp(1).

	λ_2	$\hat{d}$	$S_{\hat{d}}^2$	Z	X^2	G^2	T_n	p-value (z)	p-value (X^2)	p-value (G^2)
I	1	0.0004	0.0012	1.15	4.09	4.1	0.76	0.1251	0.767	0.766
	1.02	0.0004	0.0014	1.17	4.9	4.66	0.40	0.121	0.673	0.7
	1.03	0.0007	0.0021	1.42	8.1	7.62	0.64	0.0778	0.337	0.38
II	1.039	0.0009	0.003	1.72	12.58	11.75	0.87	0.0427	0.0862	0.11
	1.04	0.001	0.0032	1.75	13.18	12.3	0.90	0.0401	0.072	0.09
	1.041	0.001	0.0033	1.79	13.79	12.86	0.92	0.0367	0.0573	0.083
III	1.05	0.0014	0.0046	2.13	20.13	18.66	1.10	0.0166	0.0055	0.009
IV	1.06	0.002	0.0065	2.52	28.97	26.67	1.38	0.0059	0	0

7. Test of Independence Based on Mutual Information

Suppose a random sample of size n is drawn from a population. The observations in the random sample are classified according to the two criteria. Using the first criterion, each observation is associated with one of the R rows, and using the second criterion, each observation is associated with one of the C columns. Consider O_{ij} as the number of observations associated with row i and column j simultaneously. These O_{ij} s are arranged in a R×C contingency table.

Y / X	1	2	$\cdots$	C	Sum
1	O_{11}	O_{12}	$\cdots$	O_{1C}	$O_{1.}$
2	O_{21}	O_{22}	$\cdots$	O_{2C}	$O_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
R	O_{R1}	O_{R2}	$\cdots$	O_{RC}	$O_{R.}$
Sum	$O_{.1}$	$O_{.2}$	$\cdots$	$O_{.C}$	n

The total number in i-th row is $O_i.$ and j-th column is $O_{.j}$ and the sum of numbers in all the cells is n. For testing H_0: (the event "an observation is in row i" is independent of the event "that some observation is in column j" for all i and j.), by the definition of independence of events,

we can state as follows:
H_0: p(row i, column j) = p(row i) ×p(column j) for all i, j
Versus
H_1: p(row i, column j)≠ p(row i) ×p(column j) for some i, j
The test statistic is given by

$$X^2 = \sum_j \sum_i \frac{(O_{ij} - e_{ij})^2}{e_{ij}},$$

where $e_{ij} = \frac{O_{i.} \times O_{.j}}{n}$.This statistic has Chi-squared distribution with $(R-1)(C-1)$ degrees of freedom. So, reject H_0 at level α if $X^2 > \chi^2((R-1)(C-1), \alpha)$.
Now using the mutual information given in definition 2, the above testing problem is equivalent to testing the hypothesis $H_0 : I(X, Y) = 0$ versus $H_1 : I(X, Y) \neq 0$, i.e., to test $H_0 : D(p(x, y)||p(x)p(y)) = 0$ versus $H_1 : D(p(x, y)||p(x)p(y)) > 0$.
Testing the above hypothesis is similar to that of Section 4. In this case, the ML estimator of $I(X, Y)$ is given by

$$\hat{I}(X, Y) = \frac{1}{n} \sum_i \sum_j O_{ij} \log_2 \frac{O_{ij}}{e_{ij}},$$

where $O = (O_{11}, O_{12}, \ldots, O_{RC})$ is the observed values and has a multinomial distribution. For large sample size it has an asymptotic normal distribution. By using the same argument as in Section 4, we have.

$$Z = \frac{\sqrt{n}\left(\hat{I}(X, Y) - I(X, Y)\right)}{\hat{\sigma}} \xrightarrow{d} N(0, 1),$$

where

$$\hat{\sigma}^{\,2} = \frac{1}{n}\left[\sum_i \sum_j O_{ij}\left(\log_2 \frac{O_{ij}}{e_{ij}}\right)^2 - \left(\sum_i \sum_j O_{ij} \log_2 \frac{O_{ij}}{e_{ij}}\right)^2\right].$$

So, we reject H_0 if $Z_0 = \frac{\sqrt{n}(\hat{I}(X,Y))}{\hat{\sigma}} > Z_\alpha$.

The following examples introduce the use of this model of testing in comparison to the usual test of independence based on X^2-statistic (Bhattacharyya and Johnson,1977).

Example 4. 1200 persons are classified into four business groups and two types of opinion. The results are classified in the following table. Are these two types of opinion independent from each other?

opinion / group	I	II	Sum
A	32	269	300
B	51	199	250
C	67	233	300
D	83	267	350
Sum	233	967	1200

From this table, we have $X^2 = 20.59$ and by comparing it with $\chi^2(3, 0.05)$ $= 7.815$, we reject the independent hypothesis. Based on mutual information, we have $Z_0 = 2.514$ and by comparing it with $Z_{0.05} = 1.645$ we also reject the independent hypothesis. Therefore, the two methods have the same result.

Example 5. To distinguish the efficiency of a chemical treatment on seeds, we choose 100 seeds with treatment and 150 seeds without treatment and gain the following results:

	positive	negative
with treatment	84	16
without treatment	132	18

for testing independence, we have $X^2 = 0.817$ and $\chi^2(1, 0.05) = 3.841$, which imply independence. Moreover, with the new method, we have $Z_0 = 0.4456$, which is not greater than $Z_{0.05} = 1.645$. Thus, we conclude that the independent hypothesis is not rejected.

8. Conclusion

In this paper, we introduce a new statistic for goodness of fit test based on relative entropy and compare it with the classical statistics X^2, G^2 and T_n (in the continuous case) by simulation studies. It is seen that goodness of fit test based on relative entropy is more sensitive than the usual ones to the rejection of distributions which are almost close to the desired distribution. Also, to test the independence,we derive a new test based on mutual information.

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Maryam Sharifdoost
Ph.D student, Islamic Azad University
Science & Research Branch
Tehran, Iran.
E-mail: sharifdoost@gmail.com

Nader Nematollahi
Department of Statistics
Allameh Tabatabaie University
Tehran, Iran.
E-mail: na_nemat@yahoo.com

Einollah Pasha
Department of Statistics
Teacher Training University
Tehran, Iran.
E-mail: pasha@saba.tmu.ac.ir