

Third Term of the Lower Autocentral Series of Abelian Groups

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Abstract. Let G be a group and $Aut(G)$ be the group of automorphisms of G . Then $[g, \alpha, \beta] = (g^{-1}g^\alpha)^{-1}(g^{-1}g^\alpha)^\beta$ is the autocommutator of the element $g \in G$ and $\alpha, \beta \in Aut(G)$ of weight 3. Also, we define $K_2(G) = \langle [g, \alpha, \beta] : g \in G, \alpha, \beta \in Aut(G) \rangle$ to be the third term of the lower autocentral series of subgroups of G . In this paper, it is shown that every finite abelian group is isomorphic to the third term of the autocentral series of some finite abelian group.

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1. Introduction

For every element g of a group G and any automorphism $\alpha \in Aut(G)$,

$$[g, \alpha] = g^{-1}g^\alpha$$

is the *autocommutator* of the element g and the automorphism α . Clearly, if α is taken to be an inner automorphism then we obtain the commutator of two elements of G . Now the *autocommutator subgroup* of G is

defined as follows:

$$K_1(G) = \langle [g, \alpha] : g \in G, \alpha \in \text{Aut}(G) \rangle,$$

which is a characteristic subgroup of G .

There are some results on the autocommutator subgroup of a finite group G (see [2, 3, 4]). Recently, C. Chis, M. Chis and G. Silberberg ([1]) in 2008 showed that, every finite abelian group is the autocommutator subgroup of some finite abelian group. In the present paper, it is shown that a similar result holds for the autocommutator of weight three.

The following definition is vital in our investigation.

Definition 1.1. *Let g be an element of a given group G and $\alpha, \beta \in \text{Aut}(G)$. Then we define*

$$[g, \alpha, \beta] = [g, \alpha]^{-1}[g, \alpha]^\beta$$

to be the autocommutator of the element g and the automorphisms α and β of weight 3. Clearly, when α and β are taken to be inner automorphisms of G , one obtains the usual commutator of weight 3.

Now we define the third term of the lower autocentral series of subgroups as follows:

$$K_2(G) = \langle [g, \alpha, \beta] : g \in G, \alpha, \beta \in \text{Aut}(G) \rangle .$$

Note that one may define the lower autocentral series of subgroups of a given group, which are all characteristic subgroups. The main objective of this paper is to prove the following result.

Main Theorem. *Every finite abelian group is the third term of the lower autocentral series of some finite abelian group.*

2. Preparatory Results

To prove our main theorem we first establish some preparatory results. All groups, which are considered in this paper are finite.

Lemma 2.1. *Let A and B be characteristic subgroups of a given group G such that $G = A \times B$. Then $K_2(G) = K_2(A) \times K_2(B)$.*

Proof. Let $\varphi : \text{Aut}(G) \longrightarrow \text{Aut}(A) \times \text{Aut}(B)$, given by $\alpha \mapsto (\alpha|_A, \alpha|_B)$ and $\theta : \text{Aut}(A) \times \text{Aut}(B) \longrightarrow \text{Aut}(G)$, given by $(\mu, \eta) \mapsto \bar{\mu}\bar{o}\bar{\eta}$, where $\bar{\mu}\bar{o}\bar{\eta}$ is an automorphism of G , defined as follows:

$$\bar{\mu}(ab) = a^\mu b,$$

$$\bar{\eta}(ab) = ab^\eta,$$

for all $a \in A$ and $b \in B$. Since A and B are characteristic subgroups, it follows that φ and θ are inverse isomorphisms and hence we may identify $\text{Aut}(G)$ with $\text{Aut}(A) \times \text{Aut}(B)$.

Now, for any $g = ab = ba \in G = A \times B$ and $\alpha, \beta \in \text{Aut}(G)$ we have

$$\begin{aligned} [g, \alpha, \beta] &= [ab, \alpha, \beta] = [ab, \alpha]^{-1} [ab, \alpha]^\beta \\ &= (ab)^{-\alpha}(ab) \cdot (ab)^{-\beta}(ab)^{\alpha\beta} \\ &= b^{-\alpha} a^{-\alpha} ab \cdot a^{-\beta} b^{-\beta} \cdot a^{\alpha\beta} b^{\alpha\beta} \\ &= a^{-\alpha} a a^{-\beta} a^{\alpha\beta} \cdot b^{-\alpha} b b^{-\beta} b^{\alpha\beta} \\ &= [a, \alpha|_A, \beta|_B] [b, \alpha|_A, \beta|_B]. \end{aligned}$$

Thus $K_2(G) \subseteq K_2(A) \times K_2(B)$.

Now, for any $a \in A$ and $\mu, \mu' \in \text{Aut}(A)$,

$$[a, \mu, \mu'] = [a, \bar{\mu}, \bar{\mu}'] \in K_2(G).$$

Hence $K_2(A)$ is contained in $K_2(G)$. Similarly $K_2(B) \subseteq K_2(G)$ and so $K_2(G) = K_2(A) \times K_2(B)$. \square

Lemma 2.2. *Let G be a finite cyclic group. Then $K_2(G) = G^4$.*

Proof. Let $G = \langle x : x^n = 1 \rangle$ be the cyclic group of order n . Clearly φ is an automorphism of G if and only if

$$\varphi : x \mapsto x^i, \quad 1 \leq i \leq n,$$

where $(i, n) = 1$, (see [5]). Assume n is an odd number, then since the group G is abelian, the map α given by $x \mapsto x^{-1}$ is an automorphism of G . Hence, for all $g \in G$,

$$g^4 = [g, \alpha, \alpha] \in K_2(G).$$

Thus $G^4 \subseteq K_2(G)$. By the assumption $(4, n) = 1$, then there exist some integers $s, r \in Z$ such that $4s + nr = 1$. Thus $g = g^{4s+nr} = (g^s)^4 \in G^4$. This shows that G and hence $K_2(G)$ is contained in G^4 . Therefore in this case, $K_2(G) = G^4$.

Now, we assume n is even. Hence for a non-trivial automorphism of G given by $x \mapsto x^i$, the integer i must be odd and greater than 2. Therefore for any $\alpha, \beta \in \text{Aut}(G)$, with $\alpha(x) = x^i$ and $\beta(x) = x^j$,

$$[x, \alpha, \beta] = [x^{i-1}, \beta] = x^{1-i}x^{(i-1)j} = x^{(i-1)(j-1)} \in G^4.$$

Thus $K_2(G) = G^4$. \square

Lemma 2.3. *Let G be an abelian group of odd order and Z_2 the cyclic group of order 2. Then $K_2(G)$ and $K_2(G \times Z_2)$ are both isomorphic with G .*

Proof. Since G is abelian of odd order n , say, we conclude that $(4, n) = 1$. Lemma 2.2. implies that $K_2(G) = G$. On the other hand, Z_2 and G are characteristic subgroups in $G \times Z_2$. Thus Lemma 2.1. gives the assertion. \square

Lemma 2.4. *Let G be a cyclic group of order 2^m and H an abelian 2-group of exponent 2^n , with $n < m$. Then $K_2(G \times H) = G^4 \times H^2$.*

Proof. Let $G = \langle x : x^{2^m} = 1 \rangle$, then for every element $h \in H$, we may define a unique automorphism $\alpha_h \in \text{Aut}(G \times H)$ in the following way:

$$\alpha_h : (x, h_1) \longrightarrow (x, h^{-1}h_1^{-1}).$$

Thus $h^2 = [x, \alpha_h, \alpha_{e_H}] \in K_2(G \times H)$, and so

$$H^2 \subseteq K_2(G \times H).$$

Using Lemma 2.2. we have $K_2(G) = G^4$. Hence

$$G^4 = K_2(G) \subseteq K_2(G \times H).$$

So $G^4 \times H^2 \subseteq K_2(G \times H)$. On the other hand, for all $\alpha, \beta \in \text{Aut}(G \times H)$ and noting the structures of the groups G and H we have

$$[g, \alpha, \beta] \in G^4 \times H^2, \quad \text{for all } g \in G \times H.$$

Hence the result holds. \square

Using the above lemma, we obtain the following.

Proposition 2.5. *For natural numbers $m > n_1 \geq n_2 \geq \dots \geq n_r$, such that $m \geq 2$, we have*

$$K_2(Z_{2^m} \times Z_{2^{n_1}} \times \dots \times Z_{2^{n_r}}) = Z_{2^{m-2}} \times Z_{2^{n_1-1}} \times \dots \times Z_{2^{n_r-1}}.$$

3. Proof of the Main Theorem

Let G be a finite abelian group. Then G can be written as a direct product of its Sylow p -subgroups.

If $(4, |G|) = 1$, then G is of odd order and has no Sylow 2-subgroups. Since G is abelian, all of its Sylow p -subgroups are characteristic and of odd order, which by Lemma 2.3. implies that $K_2(G) = G$. Otherwise, G has a Sylow 2-subgroup, say P , and it can be written as follows:

$$P = Z_{2^m} \times Z_{2^{n_1}} \times \dots \times Z_{2^{n_r}}.$$

Now, we construct the abelian group

$$H = Z_{2^{m+2}} \times Z_{2^{n_1+1}} \times \dots \times Z_{2^{n_r+1}} \times P_1 \times \dots \times P_s,$$

where P_i 's are all Sylow p_i -subgroups of G , except P . Using Proposition 2.5, we obtain

$$K_2(H) \cong G,$$

which proves the theorem. \square

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