

Degree of Approximation and Green Potential *

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Abstract. We will relate the degree of rational approximation of a meromorphic function f to the minimum value, on the natural boundary of f , of Green potential of the weak* limit of the normalized pole-counting measures.

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1. Introduction

This paper is about the quantity

$$d = d(S, f, E) := \inf_{\{s_n\} \in S} \limsup_{n \rightarrow \infty} \|f - s_n\|_E^{\frac{1}{n}} \quad (1)$$

which we call *the degree of approximation from S to f on E* . Here, E is a subset of $\overline{\mathbb{C}}$, the extended complex plane; f is a function defined on E ; S is a class of sequences of functions; and $\|\cdot\|_E$ denotes the sup-norm on E .

One of the earliest ideas for approximation came from Sangamagrama Madhava (1350-1425). He described—all in words—how to represent the length of a circular arc as an infinite series, in terms of sine and cosine of the corresponding central angle,

$$L = \frac{r \sin \theta}{\cos \theta} - \frac{r \sin^3 \theta}{3 \cos^3 \theta} + \cdots$$

*Invited paper

for $0 \leq \theta \leq 45^\circ$, which is equivalent to what we now call the Taylor/Maclaurin series for arc tangent.

To relate this historic example to our discussion here, let $f(x) = \arctan x$, and see whether the degree of approximation from the class S of all sequences of polynomials $\{s_n\}$ (with real or complex coefficients) of respective degrees $\leq n$ is achieved by Madhava's series on a certain set E .

First, consider the case where $E = [-1, 1]$. If we let

$$s_n(x) = \sum_{k=1}^n \frac{\sin \frac{k\pi}{2}}{k} x^k \quad (2)$$

for $n = 1, 2, \dots$, then for this sequence the lim-sup in (1) would equal 1, so $d \leq 1$. But from two classical results of Bernstein ([2,16]) we can see that $d = \sqrt{2} - 1 < 1$. Bernstein's theorems show that when f is holomorphic inside an ellipse with foci ± 1 , and has at least one singularity on the ellipse, letting $\{s_n\}$ be the sequence of polynomials of best uniform approximation on $[-1, 1]$, the lim-sup in (1) equals the difference in the lengths of the semi-axes of the ellipse. In our case, the principle branch of $f(z) = \arctan(z)$ is holomorphic in $\overline{\mathbb{C}}$, minus its branch cut, which starts at $-i$, goes to ∞ along the negative imaginary axis, and then comes back along the positive imaginary axis to end at i . Therefore the largest ellipse within which f is holomorphic has a minor semi-axis of length 1, and a major semi-axis of length $\sqrt{2}$, leading to $d = \sqrt{2} - 1 < 1$, since the inf in (1) is reached through the polynomials of best approximation.

Note that when we enlarge the interval E , d moves up closer to 1, and when we shrink it, d gets smaller. It is zero when $E = \{0\}$. In any case where E is a real interval, d is strictly less than the lim-sup in (1) with s_n as in (2). For $E = [-a, a]$ with $0 < a \leq 1$, we have $d = (\sqrt{1+a^2} - 1)/a$.

Next, let $E = \{z \in \mathbb{C} : |z| \leq a\}$ where $0 < a < 1$. From a more general result of Walsh ([18, §4.6]) we can see that if the lim-sup in (1) was smaller than a for some sequence of polynomials s_n of respective degree $\leq n$, then f , the arc tangent, would have to be holomorphic inside a disc larger than the unit disc, which we know it is not the case because the arc tangent has branch points at $\pm i$. On the other hand

the sequence (2) obviously produces a as the lim-sup in (1). Therefore $d = a$ in this case.

The above situations are all about polynomials, which can be considered as rational functions with poles at ∞ . We shall see that in a more general situation we can allow the poles to converge (in some sense) to some arbitrary Borel measure whose support is not necessarily the point at ∞ . This paper is an extension of what Sergei Natanovich Bernstein, Franciszek Leja, and Joseph Leonard Walsh studied. Numerous other mathematicians have worked on this and related areas. For just a few of those results see [1,3,4,5,6,7,8,10,11,12,14,15,16].

Let f be meromorphic in D , a domain on the extended complex plane $\overline{\mathbb{C}}$. Suppose that $D \neq \overline{\mathbb{C}}$, and that the ∂D is the natural boundary of f . Let E be a closed subset of D such that f is holomorphic on E , and let $G := \overline{\mathbb{C}} \setminus E$. We assume that G has finitely many connected components, each intersecting the ∂D . Let μ be a unit Borel measure with support $Supp(\mu) \subset G$, such that it intersects every connected component of G . Let h_G^μ be the Green potential of μ in G ([9, §I.3], [13, §II.5], [16]). These are some of the properties of h_G^μ :

- (1) $h_G^\mu > 0$ in G .
- (2) h_G^μ is lower semi-continuous on $\overline{\mathbb{C}}$, superharmonic in G , and harmonic in $G \setminus Supp(\mu)$.
- (3) For quasi-every $y \in \partial G$ (for every y except a subset of logarithmic capacity 0) we have

$$\lim_{x \rightarrow y, x \in G} h_G^\mu(x) = 0. \quad (3)$$

Let the ∂G be regular, in the sense that (3) holds for all $y \in \partial G$. Let H be the union of all those connected components of $\overline{\mathbb{C}} \setminus Supp(\mu)$ that intersect the ∂G . Then we define

$$E_\sigma^\mu := \{x \in H : h_G^\mu(x) < \log \sigma\}.$$

Given rational functions r and q , we define the *pole-counting measures* π^r and κ_q^r as

$$\pi^r(A) := \text{the number of poles of } r \text{ in } A \text{ (counting multiplicities)}$$

and

$$\kappa_q^r(A) : = \text{the number of those poles of } r \text{ in } A \\ \text{that are not poles of } q \text{ (counting multiplicities)}$$

for every Borel set $A \subset \overline{\mathbb{C}}$.

We define the class S^μ as the collection of all infinite sequences $\{s_n\}$ of rational functions satisfying the following conditions:

1.

$$\limsup_{n \rightarrow \infty} \frac{\pi^{s_n}(\overline{\mathbb{C}})}{n} \leq 1.$$

2.

$$\kappa_{s_n}^{s_{n+1}}(\overline{\mathbb{C}}) = o\left(\frac{n}{\log n}\right).$$

3.

$$\frac{1}{\pi^{s_n}(\overline{\mathbb{C}})} \pi^{s_n} \xrightarrow{\text{weak}^*} \mu.$$

4. The sequence $\{\pi^{s_n}(K)\}$ is bounded for every closed set $K \subset \overline{\mathbb{C}} \setminus \text{Supp}(\mu)$.

2. Main Results

The theorems of this section are corollaries to the main theorems of [16].
Let

$$-\log \delta = \min_{x \in \partial D} h_G^\mu(x) \leq \min_{x \in \text{Supp}(\mu)} h_G^\mu(x).$$

Then it follows from [16, Theorem 4] that we can find $\{t_n\} \in S^\mu$ such that

$$\limsup_{n \rightarrow \infty} \|f - t_n\|_E^{\frac{1}{n}} \leq \delta.$$

This proves

Theorem 2.1. *If*

$$\min_{x \in \partial D} h_G^\mu(x) \leq \min_{x \in \text{Supp}(\mu)} h_G^\mu(x),$$

then

$$d(S^\mu, f, E) \leq e^{-\min_{x \in \partial D} h_G^\mu(x)}.$$

Let

$$\min_{x \in \partial D} h_G^\mu(x) < -\log \delta < \min_{x \in \text{Supp}(\mu)} h_G^\mu(x).$$

Now suppose that

$$\limsup_{n \rightarrow \infty} \|f - t_n\|_E^{\frac{1}{n}} \leq \delta$$

for some sequence $\{t_n\} \in S^\mu$. Then by [16, Theorem 3] f is meromorphic in

$$E_{1/\delta}^\mu = \{x \in H : h_G^\mu(x) < -\log \delta\}$$

which is a contradiction, since this set includes some points of the natural boundary of f . This proves:

Theorem 2.2. *If*

$$\min_{x \in \partial D} h_G^\mu(x) < \min_{x \in \text{Supp}(\mu)} h_G^\mu(x),$$

then

$$d(S^\mu, f, E) = e^{-\min_{x \in \partial D} h_G^\mu(x)}.$$

3. Some Special Cases

In this section we consider some special cases:

3.1. Let f be an entire function but not a polynomial. Therefore the natural boundary of f is $\{\infty\}$. Let E be the unit disc, and μ be the unit mass at ∞ . Then

$$h_G^\mu(x) = \max\{0, \log |x|\}.$$

The class S^μ includes the sequence of partial sums of the Taylor/Maclaurin series for f . In this case

$$d(S^\mu, f, E) = 0 = e^{-\min_{x \in \partial D} h_G^\mu(x)}.$$

3.2. Let f be holomorphic in $\mathbb{C} \setminus \{0\}$, with essential singularities at 0 and ∞ . Then the natural boundary of f is $\{0, \infty\}$. Let E be the unit circle. Let μ be a unit mass with mass $m > 0$ at 0, and mass $1 - m > 0$ at ∞ . Then

$$h_G^\mu(x) = \max \{0, \log |x|\} - m \log |x|.$$

The class S^μ includes a sequence of partial sums of the Laurent series for f , rearranged so that the poles at 0 and ∞ remain in the same proportion as the mass of μ at 0 and ∞ asymptotically. In this case

$$d(S^\mu, f, E) = 0 = e^{-\min_{x \in \partial D} h_G^\mu(x)}.$$

3.3. Let f be holomorphic in an annulus centered at the origin with radii a and b where $0 < a < 1 < b < \infty$. Let the boundary of this annulus be the natural boundary of f . Let E and μ be as in the previous case. Then h_G^μ and S^μ are as in the previous case. In this case, by Theorem 2.2.

$$d(S^\mu, f, E) = e^{-\min_{x \in \partial D} h_G^\mu(x)} = \max \{a^m, b^{m-1}\}.$$

3.4. In the next section we will construct a special function as a rational series, that is holomorphic inside the unit disc, with the unit circle as its natural boundary. Let f be that function. Let E be the closed disc of radius $a < 1$, centered at the origin, and μ be the uniform distribution of a unit mass on the unit circle. Then the class S^μ includes the sequence of partial sums of the rational series (4) which gives

$$d(S^\mu, f, E) = 0.$$

(See Property 4 of f in the next section.) On the other hand

$$h_G^\mu(x) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{a^2 - xe^{-i\theta}}{ax - ae^{i\theta}} \right| d\theta.$$

On the natural boundary of f , the unit circle, h_G^μ is constant, positive, and finite. Therefore, in this case

$$d(S^\mu, f, E) = 0 < e^{-\min_{x \in \partial D} h_G^\mu(x)}.$$

Note that in cases 3.1. and 3.2. as well as in this case, the support of μ lies on the natural boundary of f , in which Theorem 2.1. applies. This case shows that in the conclusion of the theorem strict inequality is possible in some cases.

3.5. Up to this point, we have made the assumption that f is holomorphic on E . In this special case we are replacing that condition with continuity. Let E be the closed unit disc. As in the previous case, let f be the function to be constructed in the next section, which is holomorphic in the $\text{int}(E)$ and continuous on E , with the ∂E as its natural boundary. And let μ be as in the previous case. Then, in this case

$$d(S^\mu, f, E) = 0 < 1 = e^{-\min_{x \in \partial D} h_G^\mu(x)}.$$

4. A Special Rational Series

Here, we will construct a function f in the form

$$f(z) = \sum_{k=1}^{\infty} \frac{b_k}{z_k - z} \quad (4)$$

for $z \in \Delta : |z| \leq 1$, with the properties that:

1. f is continuous on Δ ,
2. f is holomorphic in the $\text{int}(\Delta)$,
3. f has the $\partial\Delta$ as its natural boundary,
- 4.

$$\left\| \limsup_{n \rightarrow \infty} \frac{b_n}{z_n - \cdot} \right\|_{\Delta}^{\frac{1}{n}} = 0,$$

5. and every point of the $\partial\Delta$ is a limit point of the poles, in a way that

$$\frac{1}{\pi^{s_n}(\mathbb{C})} \pi^{s_n} \xrightarrow{\text{weak}^*} \mu,$$

where s_n is the n -th partial sum of (4) and μ is the uniform distribution of a unit mass on the $\partial\Delta$.

(We used this special rational series for cases 3.4 and 3.5, but it has interesting properties for its own sake.)

Let us begin the construction:

Every positive integer k can be uniquely written as

$$k = 2^{q_k} + r_k$$

where q_k and r_k are nonnegative integers and $0 \leq r_k < 2^{q_k}$. We define δ_k , θ_k , z_k , and b_k respectively as

$$\begin{aligned} \delta_k &= 2^{-2q_k} \pi^2 \\ \theta_k &= 2^{-q_k} \left(r_k + \frac{1}{2} \right) 2\pi \\ z_k &= (1 + \delta_k) e^{i\theta_k} \\ b_k &= \frac{1}{k^k} \end{aligned}$$

for each positive integer k . Clearly f is continuous on Δ and analytic in the $\text{int}(\Delta)$, so it remains to prove that the $\partial\Delta$ is the natural boundary of f .

For the following lemmas, we define ς_j , $\lambda_{j,k}$, and $\tau_{j,k}$ respectively as

$$\begin{aligned} \varsigma_j &= \frac{1}{2} e^{i\theta_j} \\ \lambda_{j,k} &= |z_k - \varsigma_j| \\ \tau_{j,k} &= \arg(z_k - \varsigma_j) - \theta_j \end{aligned}$$

for $j, k \in \mathbb{N}$, where $\arg$ is chosen so that $-\pi < \tau_{j,k} \leq \pi$.

For each $j \in \mathbb{N}$, we define $M_j \subset \mathbb{N}$ as the set of all positive integers m with the following properties:

1. $m > \max\{j, 110\}$
2. $\theta_m - \theta_j = 2^{-q_m} \pi$
3. $q_{m-2} = q_{m-1} = q_m = q_{m+1}$

Clearly M_j is an infinite set. For convenience we state the following lemma, which is a consequence of the 1st property:

Lemma 4.1. *We have*

$$\left(\frac{1}{15}\right) \left(\frac{1}{44}\right) 2^{2q_m} m \log m > 2^{q_m+1}$$

for all $m \in M_j$.

Since $\tau_{j,m} > \theta_m - \theta_j = 2^{-q_m} \pi$ or $2^{q_m+1} \tau_{j,m} > 2\pi$, and $\tau_{j,m-1} = -\tau_{j,m}$, for $m \in M_j$, the following key lemma is an easy consequence of Lemma 4.1.

Lemma 4.2. *For each $j \in \mathbb{N}$ and $m \in M_j$, there exists $n_{j,m} \in \mathbb{N}$, such that*

$$\left(\frac{1}{44}\right) 2^{2q_m} m \log m < n_{j,m} < \left(\frac{16}{15}\right) \left(\frac{1}{44}\right) 2^{2q_m} m \log m \quad (5)$$

and

$$\cos n_{j,m} \tau_{j,m} = \cos n_{j,m} \tau_{j,m-1} > \frac{1}{2}$$

hold.

Here is a sufficient condition for the main result of this section:

Lemma 4.3. *The function f defined by (4) has the $\partial\Delta$ as its natural boundary if*

$$\limsup_{n \rightarrow \infty} \left| \sum_{k=1}^{\infty} \frac{b_k \cos n \tau_{j,k}}{\lambda_{j,k}^n} \right|^{\frac{1}{n}} \geq 2 \quad (6)$$

for all $j \in \mathbb{N}$.

Proof. Since the series (4) is uniformly convergent on Δ we have

$$\begin{aligned} \left| \frac{f^{(n-1)}(\varsigma_j)}{(n-1)!} \right| &= \left| \sum_{k=1}^{\infty} \frac{b_k}{(z_k - \varsigma_j)^n} \right| \\ &= \left| \sum_{k=1}^{\infty} \frac{b_k e^{-i(\tau_{j,k} + \theta_j)n}}{\lambda_{j,k}^n} \right| \\ &\geq \left| \sum_{k=1}^{\infty} \frac{b_k \cos n \tau_{j,k}}{\lambda_{j,k}^n} \right| \end{aligned}$$

for all $j, n \in \mathbb{N}$. This completes the proof, since the set $\{\varsigma_j : j \in \mathbb{N}\}$ is dense on the circle $|z| = \frac{1}{2}$, and that already f is analytic in the $\text{int}(\Delta)$. $\square$

As a simple application of the law of cosine to the triangle with vertices z_k, ς_j , and the origin, we get the following lemma:

Lemma 4.4. *For all $j, k \in \mathbb{N}$, we have*

$$(2\lambda_{j,k})^2 = 1 + 4(\delta_k + 1)(\delta_k + 1 - \cos t_{j,k})$$

where $t_{j,k} = \theta_k - \theta_j$.

Next, consider a very useful lemma for the subsequent results:

Lemma 4.5. *We have*

$$\lambda_{j,m} < \lambda_{j,2m-3} < \lambda_{j,m+1}$$

for all $j \in \mathbb{N}$ and $m \in M_j$.

Proof. The second inequality is easily observed from the fact that

$$\theta_j - \theta_{2m-3} < \theta_j - \theta_{m-2} = \theta_{m+1} - \theta_j$$

and for the first inequality it is sufficient to show that

$$(\delta_m + 1)(\delta_m + 1 - \cos t_{j,m}) < (\delta_{2m-3} + 1)(\delta_{2m-3} + 1 - \cos t_{j,2m-3}) \quad (7)$$

due to Lemma 4.4. Note that

$$\begin{aligned} q_{2m-3} &= q_m + 1 \\ r_{2m-3} &= 2r_m - 3 \end{aligned}$$

clearly. Now recall

$$\begin{aligned} t_{j,m} &= \theta_m - \theta_j = 2^{-q_m} \pi \\ \delta_m &= 2^{-2q_m} \pi^2 = t_{j,m}^2 \end{aligned}$$

and see that

$$\begin{aligned} \delta_{2m-3} &= \frac{1}{4} \delta_m = \frac{1}{4} t_{j,m}^2 \\ t_{j,2m-3} &= -\frac{5}{2} t_{j,m} \end{aligned}$$

which can be put in (7). Therefore it is sufficient to show

$$(t_{j,m}^2 + 1)(t_{j,m}^2 + \frac{1}{2}t_{j,m}^2) < (\frac{1}{4}t_{j,m}^2 + 1)(\frac{1}{4}t_{j,m}^2 + \frac{5^2}{2^3}t_{j,m}^2 - \frac{5^4}{3 \cdot 2^7}t_{j,m}^4)$$

or equivalently

$$\frac{625}{192}t_{j,m}^4 + \frac{877}{48}t_{j,m}^2 < 15$$

which is ensured by the 1st and 2nd properties of M_j . $\square$

As an immediate consequence of Lemma 4.5. we get the following lemma:

Lemma 4.6. *For all $j \in \mathbb{N}$ and $m \in M_j$ we have*

$$\frac{b_{m+1}}{\lambda_{j,2m-3}^{n_{j,m}}} < \frac{b_m}{8\lambda_{j,m}^{n_{j,m}}}$$

for $n_{j,m}$ as defined in Lemma 4.2.

Here's another useful lemma:

Lemma 4.7. *For all $j \in \mathbb{N}$ and $m \in M_j$ we have*

$$2^{n_{j,m}}b_{2m-2} < \frac{b_m}{8\lambda_{j,m}^{n_{j,m}}}$$

for $n_{j,m}$ as defined in Lemma 4.2.

Proof. By using Lemma 4.4. we get

$$\begin{aligned} \log(2\lambda_{j,m})^2 &< 4(\delta_m + 1)(\delta_m + \frac{1}{2}t_{j,m}^2) \\ &< \frac{15}{2}2^{-2q_m}\pi^2 \end{aligned}$$

which leads to

$$\frac{n_{j,m}}{2} \log(2\lambda_{j,m})^2 < \frac{21}{22}m \log m$$

through Lemma 4.2. On the other hand we have

$$(2m - 2) \log(2m - 2) > m \log m + \frac{21}{22}m \log m + 3 \log 2$$

which completes the proof. $\square$

Next, consider

Lemma 4.8. *For all $j \in \mathbb{N}$ and $m \in M_j$ we have*

$$\frac{3}{\lambda_{j,m-2}^{n_{j,m}}} < \frac{b_m}{\lambda_{j,m}^{n_{j,m}}}$$

for $n_{j,m}$ as defined in Lemma 2.2.

Proof. By Lemma 4.4.

$$\begin{aligned} \frac{(2\lambda_{j,m-2})^2}{(2\lambda_{j,m})^2} - 1 &= \frac{4(\delta_m+1)(\delta_m+1-\cos 3t_{j,m})-4(\delta_m+1)(\delta_m+1-\cos t_{j,m})}{1+4(\delta_m+1)(\delta_m+1-\cos t_{j,m})} \\ &= \frac{16(\delta_m+1) \cos t_{j,m} \sin^2 t_{j,m}}{1+4(\delta_m+1)(\delta_m+1-\cos t_{j,m})} \\ &< (168)2^{-2q_m} \end{aligned}$$

so we can write

$$\begin{aligned} \log \frac{(2\lambda_{j,m-2})^2}{(2\lambda_{j,m})^2} &> \left(\frac{99}{100}\right) \frac{16(\delta_m+1) \cos t_{j,m} \sin^2 t_{j,m}}{1+4(\delta_m+1)(\delta_m+1-\cos t_{j,m})} \\ &> \left(\frac{99}{100}\right) \frac{(16)(0.995) \left(\frac{99}{100}\right)^2 \pi^2 2^{-2q_m}}{1+4\left(\frac{101}{100}\right)\left(\frac{1}{100} + \frac{1}{200}\right)} \\ &> (140)2^{-2q_m} \end{aligned}$$

by estimation. Therefore by Lemma 2.2. we get

$$\begin{aligned} \frac{1}{2}n_{j,m} \log \frac{(2\lambda_{j,m-2})^2}{(2\lambda_{j,m})^2} &> \left(\frac{70}{44}\right) m \log m \\ &> \log \frac{3}{b_m} \end{aligned}$$

which completes the proof. $\square$

Another lemma:

Lemma 4.9. *Let $j \in \mathbb{N}$ and $m \in M_j$. Then*

$$\lambda_{j,m-2} < \lambda_{j,k}$$

for all $k < m - 2$.

Proof. Recall that $q_{m-2} = q_m$. It is clear geometrically that the lemma is true when $q_k = q_m$. Now suppose that $q_k < q_m$. It is sufficient to prove the lemma for the case $q_k = q_m - 1$. By Lemma 4.4. we have

$$\begin{aligned} (2\lambda_{j,m-2})^2 &< 1 + 4(\delta_m + 1)(\delta_m + \frac{1}{2}(3t_{j,m})^2) \\ &= 1 + 4(\delta_m + 1)\frac{55}{10}\delta_m \end{aligned}$$

and

$$\begin{aligned} (2\lambda_{j,k})^2 &> 1 + 4(4\delta_m + 1)(4\delta_m + \frac{9}{10} \cdot \frac{1}{2}(2t_{j,m})^2) \\ &= 1 + 4(4\delta_m + 1)\frac{58}{10}\delta_m \end{aligned}$$

which completes the proof. $\square$

Now we can prove the main theorem of this section:

Theorem 4.10. *The natural boundary of f is the unit circle.*

Proof. Let j be a positive integer. Then it is sufficient to prove (6). Note that for each $m \in M_j$ we can write

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \frac{b_k \cos n_{j,m} \tau_{j,k}}{\lambda_{j,k}^{n_{j,m}}} \right| &> \frac{b_m + b_{m-1}}{2\lambda_{j,m}^{n_{j,m}}} - \frac{\sum_{k=1}^{m-2} b_k}{\lambda_{j,m-2}^{n_{j,m}}} - \frac{\sum_{k=m+1}^{2m-3} b_k}{\lambda_{j,2m-3}^{n_{j,m}}} \\ &\quad - 2^{n_{j,m}} \sum_{k=2m-2}^{\infty} b_k \\ &> \frac{b_m}{\lambda_{j,m}^{n_{j,m}}} - \frac{3}{2\lambda_{j,m-2}^{n_{j,m}}} - \frac{3b_{m+1}}{2\lambda_{j,2m-3}^{n_{j,m}}} - \frac{3}{2} 2^{n_{j,m}} b_{2m-2} \\ &> \frac{b_m}{\lambda_{j,m}^{n_{j,m}}} - \frac{b_m}{2\lambda_{j,m}^{n_{j,m}}} - \frac{3b_m}{16\lambda_{j,m}^{n_{j,m}}} - \frac{3b_m}{16\lambda_{j,m}^{n_{j,m}}} \\ &= \frac{b_m}{8\lambda_{j,m}^{n_{j,m}}} \end{aligned}$$

because of the 3rd property of M_j , and lemmas 4.2, 4.5, 4.6, 4.7, 4.8 and 4.9. To complete the proof, observe that M_j is an infinite set of positive integers,

$$\lim_{m \rightarrow \infty} \lambda_{j,m} = \frac{1}{2},$$

and

$$\lim_{m \rightarrow \infty} b_m^{\frac{1}{n_{j,m}}} = 1,$$

due to (5). $\square$

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