**Using the Outer Approximation Algorithm For Generating All Efficient Extreme Points of DEA**

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**Abstract.** Identifying the efficient extreme units in a production possibility set is a very important matter in data envelopment analysis, as these observed, real units have the best performances. In this paper, we proposed a multiple objective programming model, in which the feasible region is the production possibility set under the assumption of variable returns to scale and the objective function consists of input and output variables. As we know, by increasing the dimensions of the problem, the set of efficient points would increase as well; thus, using the multiple objective linear programming problem-solving methods in a decision set would lead to computational problems and it would be much easier to work in the outcome set instead of the decision set. In this research, we show that the efficient points in the outcome set of the suggested multiple objective linear programming problems correspond with the efficient extreme points in data envelopment analysis. An outer approximation algorithm is presented for production of all efficient extreme points in the outcome set. This algorithm provides us with the equations for all efficient surfaces. In the outcome set, this algorithm would use few calculations to produce all the extreme points. Finally, we demonstrate the presented approach through numerical examples.

**Keywords:** Data Envelopment Analysis, Multiple objective, linear programming, Outer approximation.

**1.** **Introduction**

Data envelopment analysis (DEA) developed by Charnes et al. [12] has become one of the most widely used methods in operations research/management science. A reason for this success is that DEA is a task-oriented approach and focuses on an important task: to evaluate the relative (technical) efficiency of comparable Decision Making Units (DMUs) essentially performing the same task. Based on information about existing data on the performance of the units and some preliminary assumptions, the purpose of DEA is to empirically characterize the so-called efficient frontier (surface) based on the set of available DMUs and to project all DMUs onto this frontier. If a DMU lies on the frontier, it is referred to as an efficient unit, otherwise inefficient. Efficiency evaluation is based on the data available without taking into account the decision-makers (DM) preferences. All efficient DMUs are considered equally “good”. However, if the efficient units are not equally preferred by the decision-makers it is necessary to somehow incorporate the decision-maker's judgments or a priori knowledge into the analysis. A straightforward and widely used method has been to restrict possible values of the multipliers of so-called dual DEA models. Approach is to explicitly or implicitly gather direct preference information about the desirable input and output-values of DMUs, and insert that information in a form or another into the analysis. DEA is a technique based on mathematical programming for evaluating the relative efficiency of a set of decision-making units (DMUs). The efficiency of each DMU is determined by the efficiency frontier. The units on the efficiency frontier are assumed efficient; otherwise, they are considered as inefficient. In fact, DEA sets up a production possibility set and considers its frontier as the efficient frontier made according to the non-domination condition, see, for instance DE Witte and Marques (2010).

For this approach, some ideas can be adopted from research carried out in the field of Multiple Criteria Decision Making (MCDM), especially in Multiple Objective Linear Programming (MOLP). In MCDM /MOLP, one of the key issues is to provide a decision-makers with a tool, which makes it possible to evaluate points lying on the efficient frontier. It has been shown that the MOLP and DEA models have a similar structure, see, for instance Hosseinzadeh Lotfi et al. [29]. Thus, theory and approaches developed in MOLP for evaluating solutions on the efficient frontier can also be applied in DEA.

This is important that because the dimension of the outcome set is smaller than m+s and the dimension of the decision set is n+m+s-1, generating all or portion of outcome set is expected, in general, to be less the demanding computationally than generating all or portions of the decision set. The identification of DEA efficient units under various DEA models is equivalent to the identification of the lowest input and the highest output solutions within the production possibility set for the corresponding multi-objective programming problem. The DEA-efficient DMU corresponds to the pareto efficient solution (or non-dominated solution). From this point of view, just as in the discussion of multi-objective programming, the set of all extreme points of variable returns to scale (VRS) models in DEA have significant values in the field of DEA, See, for instance Benson [7], Rockafellar [40]. In this article, we the use outer approximation algorithm for generating the set of all efficient extreme points of models DEA with VRS as proposed by Benson [8] and to do so, we use all efficient extreme points of the outcome set of the MOLP problem.

The organization of this paper is as follows, in section 2, we present literature review. In section 3, we present MOLP problem and its relation to models DEA with VRS, and we provide the theoretical foundation of the outer approximation procedure. We summarize some relevant results concerning efficient extreme points of the MOLP problem. Section 4, provides a detailed statement of the algorithm; additionally, a small example problem is solved for illustration purpose. Section 6, provides a computational experiment and statistical analysis. Some concluding remarks are given in the last section.

**2. Literature review**

In recent years, there have been a number of studies discussing the relationship between DEA and MOLP models. In their article, Doyle and Green [20] showed that DEA is an MCDM method. Alene et al. [1] used MOLP problem solving methods to apply the decision maker’s a priori knowledge in DEA problems. Golany [26] presented a data envelopment analysis model with a MOLP structure and used interactive MOLP methods to solve the model. Their model helped the decision maker (DM) to allocate a set of inputs, such as resources, on the efficiency frontier based on the level of outputs. Joro et al. [34] revealed that DEA problems have a similar structure to MOLP models; therefore, to solve the DEA models, we can use the corresponding reference point models in MOLP.

Wong et al. [46] proposed an equivalent model between DEA and MOLP and demonstrated how to solve a DEA problem interactively, without any prior judgment, by transforming a MOLP formula. Using interactive MOLP methods, they searched for the most preferred solutions (MPS) points on the efficiency frontier along with resource allocation and target setting according to the DM’s a priori knowledge; then, they used interactive approaches such as G-D-F, Steam and Stom to solve the model and finally, engaged in a comparison of results. Yang et al. [47] attempted to demonstrate the use of interactive MOLP methods for target setting in DEA and illustrated the relationship between output-oriented DEA dual models and formulation of maxmin preferred points in MOLP models; they used the interactive projected gradient approach to identify the efficient units. Malekmohammadi et al. [39] focused on the topic of target setting in DEA using MOLP problems; they extended the models presented by Yong et al [47] to simultaneously reduce the final inputs and increase the final outputs and showed that instead of solving models, we can set our targets according to the DM’s preferences by solving only one model.

Hosseinzadeh et al. [28] evaluated the relationship between output-oriented dual models in DEA and MOLP models. In their study, they showed how a DEA model can be solved interactively by transforming a MOLP formula; in this regard, they used the Z-W approach to apply the DM’s a priori knowledge in the performance process. Ebrahimnejad et al. [21] proposed an interactive MOLP method to identify the target units in DEA models in the presence of undesirable outputs; they extended the relationship between BCC models and the reference point model in MOLP toward a simultaneous and interactive increase in desirable outputs and decrease in undesirable final outputs based on MOLP models.

The main purpose of MOLP problems is to find the set of efficient solutions. These solutions are Pareto optimal solutions that can simultaneously optimize all objective functions. Among these units, the efficient extreme units are the most important ones; these would be observed, real units and their performance would determine the performance of other units in the system.

We may search for solutions also on the efficient frontier in DEA. Since the outcome set has a much simpler structure and smaller size than the decision set, a handful of researchers in recent years have begun to turn their attention to the mathematics and tools for generating all or portions of the efficient outcome set, rather than the efficient decision set, for the MOLP problem. See, for instance, Banker et al. [2], Banker et al. [3], Benson [4], Benson and Sayin [10], Dauer and Liu [17], Dauer and Saleh [18], Dauer and Gallagher [16], Dauer [14, 15].

Various methods have been presented for identification of these units, out of which we can mention the approaches proposed by Chon [13], Evans [22], Goicoechea et al. [25], Luc [38], Sawaragi et al. [41], Steuer [43], Yu [48] and Zeleny [51].

One of these approaches is the vector maximization method, see Kuhn and Tucker [37]. We can use this method to determine all efficient points in a decision set, see Benson [5], Isermann [31], Bitran [11], Villarreal and Karwan [45], Kostreva and Wiecek [36]. The problem with all those methods of determining efficient points in the feasible region and the decision set was too many calculations and the presented approaches were not convergent in most cases; in this relation, when the problem’s dimensions, variables and constraints increase, the set of efficient points would expand in MOLP problems and we would face a difficult process for finding the efficient set. Since the outcome set has a simpler form and a smaller region compared to the decision set, it would be easier to find the efficient points in the outcome, see Steuer [44], Dauer and Liu [17], Dauer and Saleh [18], Benson [4, 6], Dauer [14, 15], Gallagher and Saleh [24], Dauer and Gallagher [16], Horst et al. [30].

Therefore, instead of directly solving the DEA models, we present a DEA model with a MOLP structure and use the MOLP model’s outcome set to specify the efficient extreme units. To find the set of efficient units in the outcome set, we can employ methods such as the outer approximation algorithm [8] and the weight set decomposition algorithm [9]. In this research, we make use of the outer approximation algorithm, which is a convergent algorithm using little calculations based on linear searching and linear programming techniques. The following methods have been proposed to find DEA efficient points using efficient surfaces in MOLP.

Jahanshahloo et al. [33] presented a method for finding the piecewise linear frontier of the production function in data envelopment analysis. Korhonen [35] introduced another method to search for the efficiency frontier in DEA. In another study, Jahanshahloo et al. [32] proposed an approach for finding strongly efficient hyperplanes of the production possibility set (PPS) in data envelopment analysis. Sayin [42] presented an algorithm for determining efficient faces in DEA. Hosseinzadeh et al. [27] proposed a new method for finding the set of efficient surfaces in DEA based on MOLP models; in this relation, they introduced a linear programming model that could find the efficient defining hyperplanes of the production possibility set.

The approach proposed in the present paper is a new and distinguished method comparing to previous approaches. The advantage to our approach is that this method can determine all efficient extreme points of the production possibility set in the outcome set through little calculations.

**3.** **Structural Similarities between MOLP and DEA**

Assume that we have observed decision-making units as , where each consumes an -vector input to produce an -vector output. Suppose that and are the vectors of inputs and outputs, respectively, for , in which it has been assumed that , and , . is the reference weight for . represents the input variable vector. shows the output variable vector. We define the production possibility set of data envelopment analysis with VRS as follows:

, , , , .

We must note that the set includes all input and output vectors (X, Y) that apply to the set’s constraints.

**Definition 3.1**  *is called an efficient point if and only if there is not an such that and*

**Definition 3.2**  *is called a weak efficient point if and only if there is not an such that .*

Consider the following MOLP problem

(1)

where is a matrix, , represent the multiples of the -th objective function in the MOLP problem. shows the Euclidean space. is the technology matrix including all variable multiples in problem (1). is an matrix, and rank()=, . , represents the decision-making variable vector in the MOLP problem and shows the feasible region of the MOLP problem.

The Pareto solution and weak Pareto solution of are defined as follows:

**Definition 3.3** * is called a Pareto solution of  if there does not exist  such that , .*

**Definition 3.4** * is called a weak Pareto solution of  if there does not exist  so that .*

Put , , ,,  ,  is a vector whose th element is one and other elements are zero, , and

,

. , , .

Then problem  is converted to

 (2)

In model (2), vector is the variable vector for inputs and outputs; we can obtain the values of this vector by solving model (2).

Note  is a feasible solution of problem (2) while  is a vector belong to objective function space of problem (2).

By considering definition 3.3  is called a pareto solution of  if there does not exist  such that  and .

**Theorem 3.1.** Let *,* then

(i)  is a Pareto solution of (2) if and only if  is an efficient unit in .

(ii)  is a weak Pareto solution of (2) if and only if  is a weak efficient unit in .

**Proof**.  Let  be a Pareto solution of . We show that  is an efficient unit in . By contradiction, suppose  is not an efficient unit in , then there is an  such that  and . Since , there is a  such that  is a feasible solution of  . Since  and , then we have a contradiction; therefore, is an efficient unit in .

Now suppose  is an efficient unit in . Since , there is a  such that is a feasible solution of . As  is an efficient unit in , there is no  such that  and . Since, there is a vector  for each  such that  is a feasible solution of . Regarding the above relations there is no  that is a feasible solution of  such that  and . Therefore is a Pareto solution of  and the proof is completed.

(ii) Proof is similar to (i).

**Theorem 3.2.** *Let ,  is called the outcome set for MOLP) then dim (*

**Proof**. Since  and , , ,  Rank and , then dim .

**Theorem 3.3.** *The optimal values of problem (2) are finite.*

**Proof**. Since,, and , , , then  are finite. Similarly , , and , , , then  are finite, Therefore, the optimal values of problem  are finite.

By using the observed DMUs, For each  and  , we put

.

.

Vector  is called the anti-ideal point of outcome set for problem . Let  satisfy , we define as follows:

, for some .

**Theorem 3.4.** *Set  is a nonempty, bounded polyhedron in  of dimension .*

**Proof**. Since , , by Theorem (3.3), the definition of  implies that  is a nonempty, bounded set in . We may write , where  and , Since , ,  are polyhedral sets as proposed by Dauer and Gallagher [10], therefore  is a polyhedral set. Since  then  show interior points set of V), by Theorem (3.2) the dimension of  is , and the proof is complete.

A point  is called an efficient (or admissible) point of  when no  exist such that  and . When no  exist such that , then  is called a weakly efficient (or weakly admissible) point of . Let  and  denote the set all efficient and weakly efficient points, respectively of .

**Theorem 3.5.** *Let  be the set of efficient point of  then .*

**Proof**. Suppose that  but  does not belong to , then by the definition of , there exists a point  such that . Since , there exists a point  such that , therefore . This contradicts , therefore .

Now suppose , to show that , we show that  for some  and (efficient points of outcome set for MOLP). Since , Therefore  for some , since  and  then . Let  satisfy , , then by the definition of , since , we have , then  does not belong to , but this is a contradiction, therefore, .

Let

 (3)

By Theorem (3.3),  is a finite number. If  is an optimal solution of , then  is an efficient solution of (we solve problem  by the weighted-sum problem method (by choosing ), see, Zeleny [48, 49]).

For , we put , and such that,  for  and

 for .

**Theorem 3.6.** *The Convex hull of  is an -dimensional simplex and contains .*

**Proof**. First we show that  is a affinely independent. Since  for  and  for ,, we put , therefore, (by the definition of , this is evident), Let  then ; since , therefore, ,  and  is a affinely independent.

To show that the convex hull contains , suppose . Since  is a Linearly independent set, hence it is a basis for ; therefore, there is ,  such that . If , we have , but we have , which contradicts the previous paragraph. Hence,  and  , therefore, (S is the convex hull of . We showed that .

**Theorem 3.7.** * may also be written as following.*

=

**Proof**. Suppose that , then



Therefore . On other hand, we have , therefore 

Now suppose that  Let , , , , then  We have , but , therefore 

By the definition of , , we have

== =, then , therefore .

**Theorem 3.8.** *Let  and suppose ,  does not belong to , and , where  is the solution of problem (4) then .*

 (4)

**Proof**. Suppose  does not belong to . Then we may choose a point  such that . Since , then , on other hand, we have  then .

Put , },

. Choose  such that  and . Let . Suppose  then , . Since  then  and  therefore , . Therefore . Similarly we show that , . We conclude that . Since , this contradicts the fact that  belong to the boundary of (consider problem (4)) so the proof is complete.

From Rockafellar [40] and Yu [48] and the weighted-sum problem,  is a face of  if and only if  is equal to the set of the optimal solution set of following problem

 (5)

for some .

The variable vector expresses the corresponding multiples of the output and input vector .

We know that  is weak efficient unit if and only if the optimal value of the following problem is zero. (It is the clear).  (6)

The dual of the linear program (6) is as follows:

 (7)

By the duality Theorem of linear programing, since the optimal value of (6) is zero, problem (7) also has an optimal value of zero, therefore

 and , , From Falk and Hoffman [23], we know that the optimal values of problem (5) correspond to weakly efficient faces of  for

. Inequality , construct inequality cuts needed in the outer approximation algorithm for generating all efficient extreme points of .

## 4. Generating All Efficient Extreme Points of the production possibility set

We apply the outer approximation algorithm for generating all efficient extreme points of the outcome set of problem (2). In what follows, all efficient extreme points of the production possibility set of the DEA with VRS are essentially immediately available upon termination of the algorithm, by converting points  in the outcome set of problem (2) to equivalent points  in .

**The Outer Approximation Algorithm** applied as follows:

**Initialization step**. Compute a point .  may be set equal to any strict convex combination of  and , where  is any optimal solution to the linear program (6) with  and construct the -dimensional simplex  containing  described in Theorems (3.6) and (3.7). Set is a -dimensional simplex consisting of the vertices of , as described in Theorems (3.6).

Store both the vertex set  given in Theorem (3.6) inequality representation of  given in the Theorem (3.7). Set  and go to iteration . Iteration , , See Steps  through  below.

**Step 1)**. If, all vertexes of  belong to , then stop . Otherwise, choose any vertex of  such that, it does not belong to , for example  and continue (To test a given  is membership in , one may apply the phase-I procedure of the simplex method to problem (6) by putting .

**Step 2)**. Compute  description in the Theorem (3.8) by putting

.

**Step 3).** Set  where

 is any dual optimal solution to the linear programing (6) with  that have been calculated in the step (2).

**Step 4)**. Using vertexes of  and method that it supposed by Falk and Hoffman [23] and definition of  given in Step (3), determine all vertexes of  , set  and go to iteration k.

By definition of  in step (3) since  don’t belong to  and , we conclude that algorithm generates distinct polyhedral ,  so that

. This implies that the algorithm must be finite and it must terminate in some iteration . is a -dimensional simplex including the vertices of , formed in each stage .

**Theorem 4.1.** *Let  denote the iteration number in which  and the outer approximation algorithm terminate. Let*

 belong to vertexes set of  and 

then  is identical to the set of all efficient extreme points of .

**Proof**. From before we have

. And, for , . is any dual optimal solution to the linear programing (6) with  in step 3). Notice also that .

Suppose that , then  are belong to vertex set of  and

, therefore at last,  of the inequalities below, must hold as equations at . , . That implies that . We show , by contradiction, suppose  dose not belong to , therefore, there is  such that , .

Let , , , .

For , let  and for , let .

We choose  such that ,  and define , for  and , for  and , for  and , for . Then , for , , and , for , and , for .

Then ,  therefore . We have  then  is a strict convex combination of  and , that is contradiction (because  is belong to vertex set of ). Therefore , from Theorem (3.5), we have . Since  then  is an efficient extreme point of .

Now suppose that  is a efficient extreme point of  and  don’t belong to all efficient extreme points of , therefore, we choose  and ,  so that . By solving problem (2) by weighted-sum problem method(see, Zeleny [49]), and from Theorem (3) in Benson [7], we conclude that  is an efficient extreme point of the polyhedron .

We may select a point

, , such that  is the unique optimal solution to the following problem

 (8)

From the definition of , this implies that  is also the unique optimal solution to the problem (8). Since ,therefore

 and

.

Since  these inequalities imply that

.

Since  the left-hand-side of the previous inequality equals , yielding a contradiction. Therefore,  belong to all efficient extreme points of  must be true, so that the proof is complete.

## 5. Application and discussion

In this section, we illustrate the problem by two numerical example.

**Example 1**.

Consider the case where there are seven units with an input and an output whose details have been given in the following Table.

Table 5.1 Input and output of the seven DMUs.

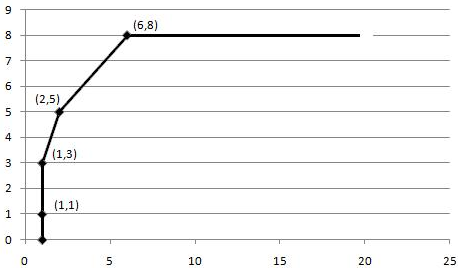
|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
| Input | 1 | 1 | 2 | 4 | 5 | 6 | 3 |
| Output | 1 | 3 | 5 | 6 | 7 | 8 | 2 |
|  | 1 | 1 | 1 | 0.83 | 0.93 | 1 | 0.33 |
| Efficiency status | non-extreme | extreme | extreme | - | - | extreme | - |

The corresponding MOLP is

 (9)

Where the efficiency frontier of production possibility set of above example is shown in Figure 5.1.

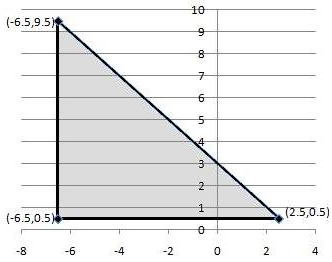
In the initialization step of the algorithm, we have ,  then =. We select  By definition of , we have , for some  belonging to the feasible region of problem (9)}.



*Figure 5.1*: The efficiency frontier of production possibility set.

Put . If we solve problem (3), we would obtain .

Therefore , . As shown in Figure 5.2.



*Figure 5.2*: The set in  space.

The vertexes set of  given in Theorem (3.6) are 

which are computed as follows:

 . Therefore  and .

In step (1) of the algorithm, since (-6.5,9.5) does not belong to , we put

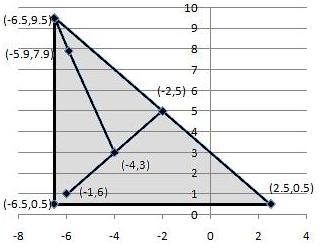
. We go to step (2). If we solved problem (6) by == we would obtain . We put, then . Now, we solved problem (4), we put  and , we would obtain  and .

As shown in Figure 5.3. We solved problem (7) by , we would obtain  , then the inequality cut is as follows.

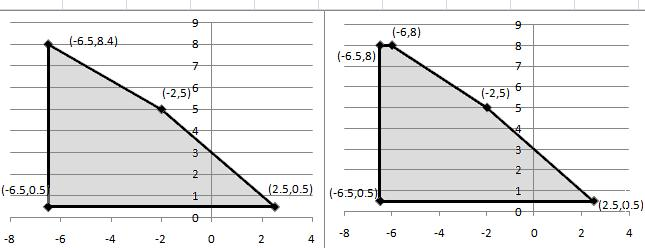
.

We go to step (3) and we organize As shown in Figure 5.4.

The vertex set of  given in the Theorem (3.6) are We put . Since (-6.5,8.35) does not belong to , we put .



*Figure 5.3*: , , and  in  space.



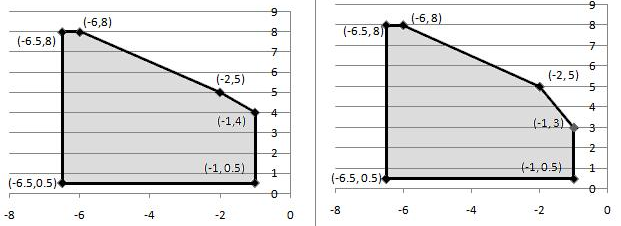
*Figure 5.4*: The  and  sets in  space.

In the next step, we solved problem (4), we put  and , we would obtain  and . We solved problem (7) by , we would obtain , then the inequality cut is as follows.

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We go to step (3) and we organize . As shown in Figure .5.4. The vertexes set of  given in Theorem (3.6) are

. We put . Since (2.5,0.5) does not belong to , we put .



*Figure 5.5*: The  and  set in  space.

In the next step, we solved problem (4), we put  and , we would obtain  and . We solved problem (7) by , we would obtain , then the inequality cut is as follows.

.

We go to step (3) and we organize . As shown in Figure 5.5. The vertexes set of  given in Theorem (3.6) are

. We put . Since (-1,4) does not belong to , we put .

In the next step, we solved problem (4), we put  and , we would obtain  and . We solved problem (7) by , we would obtain , then the inequality cut is as follows.

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We go to step (3) and we organize . As shown in Figure 5.5. The vertexes set of  given in Theorem (3.6) are



Since (-6.5,0.5), (-1,0.5) and (-6.5,8) have the same components to , then they do not belong to . Therefore, the vertexes set of  are as follows.

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We obtain the vertexes set of  by converting  to  as follows:

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**Example 2**.

Consider the case where there are five units with an input and two outputs whose details have been given in the following Table.

Table 5.2. The input and outputs of the first DMUs.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
| Inputs | 1 | 1 | 1 | 1 | 1 |
| Output1 | 6 | 5 | 2 | 3 | 2 |
| Output2 | 2 | 3.5 | 5 | 3.5 | 2 |
|  | 1 | 1 | 1 | 0.833 | 0.5 |
| Efficiency status | extreme | extreme | extreme | - | - |

The corresponding MOLP is

 (10)

In the initialization step of the algorithm, we have , ,  then =. We select  By definition of , we have , for some  belonging to the feasible region of problem (10)}.

Put . If we solve problem (3), we would obtain .

Therefore , .

The vertexes set of  given in Theorem (3.6) are 

which are computed as follows:

 Therefore  and .

In step (1) of the algorithm, since does not belong to , we put

. We go to step (2). If we solved problem (6) by ==, we would obtain . We put, then . Now, we solved problem (4), we put  and , we would obtain  and .

We solved problem (7) by , we would obtain , then the inequality cut is as follows.

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We go to step (3) and we organize

The vertex set of  given in the Theorem (3.6) are 

We put . Since does not belong to , we put .

In the next step, we solved problem (4), we put  and , we would obtain  and . We solved problem (7) by , we would obtain , then the inequality cut is as follows.

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We go to step (3) and we organize

The vertexes set of  given in Theorem (3.6) are .

We put . Since does not belong to , we put .

In the next step, we solved problem (4), we put and, we would obtain  and. We solved problem (7) by, we would obtain then the inequality cut is as follows.

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The vertexes set of  given in Theorem (3.6) are .

We put . Since does not belong to , we put .

In the next step, we solved problem (4), we put  and, we would obtain  and . We solved problem (7) by , we would obtain then the inequality cut is as follows.

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The vertexes set of  given in Theorem (3.6) are 

Since  has the same components to , then they do not belong to . Therefore, the vertexes set of  are as follows.

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We obtain the vertexes set of  by converting  to  as follows:

.

**6. Computational Experiment and statisical analysis**

To conduct a preliminary computational experiment for our proposed approach, we can use the preliminary VS-Fortran code to execute the outer linear approximation algorithm, see Benson [8]. The Horst-Thoai–De Vries method [29] is used to execute the fourth step of the algorithm; the linear bisection method is used for our univariate search in the second step, and to solve the linear programming problem, we use the simplex algorithm; as implemented by the subroutines of IMSL. [51]. Benson [8] has provided the number of iterations and efficient extreme points and the CPU introduction times for thirty multiple objective linear programming problems with different dimensions. In the present research, we use the Gams software to solve our DEA models and the Lindo software is used for solving the linear programming problems. Note that in order to determine the efficient extreme points using traditional DEA models, we need to solve at least models, which is difficult to do; it would also be quite difficult to obtain information related to the efficient surfaces. However, in this article, we arrive at all the efficient extreme points by solving only one MOLP model, and the model is not dependent on the unit under evaluation. The m+s+n model is variable in the decision set and the number of m+s is variable in the outcome set. Now, the outcome set is smaller and we can convert the efficient extreme points in this set to efficient extreme points in the decision set through a simple calculation; thus, using the presented algorithm, we can determine the efficient extreme points of the production possibility set and its efficient surfaces by solving one model and a few iterations of the algorithm. In the numerical example provided, we use the model to evaluate seven decision-making units (DMUs) under VRS technology, each having one input and one output. In the one example, there are 9 variables in the decision set and 2 variables in the outcome set; we obtained all efficient extreme points after four iterations of the algorithm. In the third step of the algorithm, we solve a linear programming problem to find the optimal values of , and to form the cutting-plane equations, a linear programming problem is solved in each stage. As can be observed, this method involves fewer calculations for finding the extreme points compared to traditional DEA models, which require solving models for the same purpose. For the example 2, we have similar interparation. The statisical analysis of examples discriped in Table 6.1.

Table 6.1. The statisical analysis of examples.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Example | The number of  Inputs and outputs | The number of  DMUs | The number of variables in feasible region | The number of efficient extreme points  in feasible region | The number of efficient extreme points in outcome space | The number of  algorithm iterations | The number of  Solved LPs |
| Example 1 | 2 | 7 | 9 | 84 | 3 | 4 | 9 |
| Example 2 | 3 | 5 | 8 | 70 | 3 | 5 | 12 |

The presented algorithm has many useful computational advantages to previous approaches for determining the efficient extreme points of the production possibility set:

1. Since the algorithm produces all efficient extreme points of the outcome set based on the decision set and the outcome set is smaller than the decision set, fewer calculations are needed for finding these points.
2. The proposed algorithm is precise and finite; thus, through solving one MOLP model and a number of iterations, we can arrive at all efficient extreme points and efficient surfaces.
3. This algorithm does not face the issues of previous algorithms in producing the efficient extreme points, such as infeasibility and degeneracy.
4. The presented approach makes a new connection between DEA and MOLP problems; in this regard, we can identify all efficient surfaces by solving one MOLP problem and multiple iterations of the algorithm.
5. The presented approach can be a new method for obtaining all efficient extreme points.

## 7. Conclusion

The purpose of this paper was to develop a new method for generating efficient extreme points

of the production possibility set with VRS. We proposed an MOLP problem whose feasible region same of is production possibility set. We applied the outer approximation algorithm for generating the efficient extreme points of MOLP problem. Since the average number of efficient extreme points in the outcome set is less than the average number of efficient extreme points in the decision set, the method proposed is pretty fit. We obtain the efficient frontier by solving an MOLP problem, the outer approximation algorithm can be implemented relatively easily by using search methods, linear programming techniques, and any one of several special methods from the global optimization literature for generating new vertexes set as linear inequality cuts are added to the containing polyhedral generated by the algorithm. This algorithm used few calculations to produce all the extreme points. The results of this paper can be used on various DEA-related application problems. The results proposed a way for extending the analysis of production efficiency to further path. In this method, we obtain the efficient frontier by solving an MOLP problem.

For future research, we suggest extending our presented approach to determine the units’ return to scale class; furthermore, we can solve the proposed MOLP problem using other methods for obtaining extreme points, such as vector maximization and make a comparison of results.

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