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Application of Homotopy Perturbation Method for Solving Hybrid Fuzzy Differential Equations

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Abstract. The hybrid fuzzy differential equations (HFDEs) are natural way to model dynamic systems with embedded uncertainty and have a wide range of applications in science and engineering that make them useful. The present study is an attempt to obtain the numerical solutions of HFDEs through homotopy perturbation method (HPM). The convergence of the HPM to solve HFDEs is investigated in detail in this paper. In addition, the validity and efficiency of the proposed method is investigated and verified by several numerical examples.

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1. Introduction

The concept of fuzzy sets, the corresponding fuzzy operations and some properties were introduced by Zadeh [35]. Many efforts have been accomplished to the development of various concepts of fuzzy theory and

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applications of fuzzy systems, in particular to the theory of differential equations. Fuzzy differential equations were first formulated by Kaleva [15] and Seikkala [31]. In [15] the properties of differentiable fuzzy sets based on the Hukuhara differentiability concept, are explained. In [31] the Seikkala fuzzy derivative is defined. They proved that the fuzzy initial value problem $x'(t) = f(t, x(t)), x(0) = x_0$ has a unique fuzzy solution provided f satisfies the Lipschitz condition. In [4] characterization theorem has been stated that a fuzzy differential equation is equivalent to a system of ordinary differential equations (ODEs). By this theorem we can numerically solve systems of ODEs instead of solving fuzzy differential equations. Note that these equations may be solved by any suitable numerical method for ODEs [4, 13, 16, 18, 21].
A hybrid system is a dynamic system that contains both continuous and discrete dynamic event. These continuous and discrete dynamics interact and coincide. These behaviour lead to dynamics as described by differential or difference equations in time. Dynamic systems are studied in at least three branches. In computer science, a hybrid system primarily is considered as a discrete computer program. The modeling and simulation branch looks at a hybrid system as physical system. Another is the systems and control branch that study a hybrid system from different aspects [30].
The differential equations containing fuzzy valued functions and interaction with a discrete time controller are named as HFDEs [26]. Analytical results on HFDEs are stated in [19, 29, 30]. In [25] by applying Bede's characterization theorem piecewise over each $[t_k, t_{k+1}]$, the characterization theorem for HFDEs has been proved. Various numerical methods have been applied for solving HFDEs. Some of these methods under Hukuhara differentiability concept such as the fuzzy Euler method, Rung-Kutta method, Nystrom method and etc. are presented by many authors. For example, Pederson and Sambandham have obtained the numerical solutions of these equations by using the fuzzy Euler and Runge-Kutta methods [25, 26, 27]. Also Allahviranloo and Salahshour used the fuzzy Euler methd to investigate approximate solutions [2]. Prakash and Kalaiselvihave studied the fuzzy predictor-corrector method [28]. Solaymani Fard and Aliabdoli Bidgoli have solved HFDEs by Nystrom method

[33]. Kim and Sakthivel used the improved predictor-corrector method to modify the numerical solutions of HFDEs [17]. Recently, Paripour et al. used Adomian decomposition method (ADM) [24], Solaymani Fard and Aliabdoli Bidgoli applied Chebyshev wavelets approximations [32], Ahmadian et al. employed a numerical algorithm based on the high order Runge-Kutta method [1] and Otadi and Mosleh applied homotopy analysis method (HAM) to investigate the series solution of the HFDEs [23].

In the present paper, the fuzzy HPM is successfully used to obtain approximate solutions of HFDEs. The HPM was established by Ji-Huan He [11]. The HPM is an universal nonlinear analytical method which has been applied to solve various linear and nonlinear functional equations in scientific and engineering [3], [7]-[12]. In this method is introduced a homotopy parameter p, which takes values from 0 to 1 and the solution is considered as the sum of an infinite series, that usually converges rapidly to exact solutions when p approaches to one [6].

This paper is divided in to the following sections. In Section 2., We begin by introducing the fuzzy number, fuzzy Hukuhara derivative and characterization theorem. In Section 3., definition of HFDEs and characterization theorem for HFDE are stated. Section 4., describes the fuzzy HPM to investigate numerical solutions for HFDEs. In Section 5. the convergence of the HPM to solve HFDEs is studied. Section 6., contains some examples, the numerical results and discussion. Finally, conclusions are made in Section 7..

2. Preliminaries

In this section, we introduce some definitions and the necessary notations which will be used throughout this paper.

Definition 2.1. [19] *A fuzzy number is a function* $u : \mathbb{R} \longrightarrow [0, 1]$ *satisfying the following properties:*

- *(i) u is normal, that is, there exists an* $x_0 \in \mathbb{R}$ *such that* $u(x_0) = 1$*;*
- *(ii) u is fuzzy convex, that is, for* $x, y \in \mathbb{R}$ *and* $0 \leqslant \lambda \leqslant 1$ *,* $\min\{u(x), u(y)\} \leqslant u(\lambda x + (1 - \lambda)y)$*;*

- *(iii) u is upper semicontinuous on* $\mathbb{R}$*;*
- *(iv)* $\overline{\{x \in R : u(x) > 0\}}$ *is compact, where* $\overline{A}$ *denotes the closure of* A*.*

The set of all fuzzy real numbers is denoted by E*.*
Another definition for a fuzzy number is the following:

Definition 2.2. [20] *A fuzzy number in parametric form is represented by an ordered pair of functions* $(\underline{u}(r), \overline{u}(r))$*, for every* $0 \leqslant r \leqslant 1$*, satisfying the following requirements:*

- $\underline{u}(r)$ *is a bounded, left continuous and nondecreasing function over [0,1].*
- $\overline{u}(r)$ *is a bounded, left continuous and nonincreasing function over [0,1].*
- $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.

A crisp number r is simply represented by $\underline{u}(r) = \overline{u}(r) = r$, $0 \leqslant r \leqslant 1$. For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and an arbitrary crisp number k we define fuzzy addition and scalar multiplication as

- $(\underline{u+v})(r) = (\underline{u}(r) + \underline{v}(r))$.
- $(\overline{u+v})(r) = (\overline{u}(r) + \overline{v}(r))$.
- $(\underline{ku})(r) = k\underline{u}(r)$, $(\overline{ku})(r) = k\overline{u}(r)$, $\quad k \geqslant 0$.
- $(\underline{ku})(r) = k\overline{u}(r)$, $(\overline{ku})(r) = k\underline{u}(r)$, $\quad k < 0$.

For every $0 < r \leqslant 1$ denote $[u]^r = \{t \in \mathbb{R} : u(t) \geqslant r\}$ and $[u]^0 = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$. Then by Definition 2.1 for every $u \in E$, the r-level set $[u]^r$ is a nonempty compact interval for all $0 \leqslant r \leqslant 1$. The notation $[u]^r = [\underline{u}^r, \overline{u}^r]$ denotes the r-level set of u. Note that $\underline{u}^r$ and $\overline{u}^r$ are as the lower and upper branches on u, respectively.
A metric in E is define by the equation

$$D(u, v) = \sup_{0 \leqslant r \leqslant 1} d([u]^r, [v]^r),$$

where d is the Hausdorff metric for nonempty compact sets in $\mathbb{R}$.
For $x, y \in E$ if there exists a $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$.

Definition 2.3. [14] *A mapping $f : I \longrightarrow E$ is differentiable at $t \in I$ if there exists a $f'(t) \in E$ such that the limits*

$$\lim_{h \longrightarrow 0^+} \frac{f(t_0 + h) - f(t_0)}{h} \quad \textit{and} \quad \lim_{h \longrightarrow 0^-} \frac{f(t_0) - f(t_0 - h)}{h},$$

exist and are equal to $f'(t_0)$. Here the limits are taken in the metric space (E, D).
The fuzzy set $f'(t_0)$ is called the Hukuhara derivative of f at t_0.
Next, we review the Bede's characterization theorem.

Theorem 2.4. [4] *(Characterization Theorem) Consider the fuzzy initial value problem*

$$\begin{cases} x'(t) = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $f : I \times E \longrightarrow E$ is such that:

- $[f(t, x)]^r = [\underline{f}^r(t, \underline{x}, \overline{x}), \overline{f}^r(t, \underline{x}, \overline{x})]$,
- *$\underline{f}^r$ and $\overline{f}^r$ are equicontinuous and uniformly bounded on any bounded set,*
- *there exists an $L > 0$ such that*

$$|\underline{f}^r(t, x_1, y_1) - \underline{f}^r(t, x_2, y_2)| \leqslant L \max\{|x_2 - x_1|, |y_2 - y_1|\}, \quad r \in [0, 1],$$

$$|\overline{f}^r(t, x_1, y_1) - \overline{f}^r(t, x_2, y_2)| \leqslant L \max\{|x_2 - x_1|, |y_2 - y_1|\}, \quad r \in [0, 1].$$

Then, the equation (1) and system of ODEs

$$\begin{cases} (\underline{x}^r(t))' = \underline{f}^r(t, \underline{x}^r, \overline{x}^r), \quad \underline{x}^r(t_0) = \underline{x}^r_0, \\ \\ (\overline{x}^r(t))' = \overline{f}^r(t, \underline{x}^r, \overline{x}^r), \quad \overline{x}^r(t_0) = \overline{x}^r_0, \end{cases} \tag{2}$$

are equivalent.

3. Hybrid Fuzzy Differential Equations

Consider the HFDE [19]:

$$\begin{cases} x'(t) = f(t, x(t), \lambda_k(x(t_k))), \; t \in [t_k, t_{k+1}], \; k = 0, 1, 2, \ldots, \\ x(t_k) = x_k, \end{cases} \tag{3}$$

where $f \in \mathcal{C}[\mathbb{R}^+ \times E \times E, E]$ and $\lambda_k \in \mathcal{C}[E, E]$. Suppose that $0 \leqslant t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ are such that $t_k \longrightarrow \infty$ as $k \longrightarrow \infty$ and the existence and uniqueness of solutions of the hybrid systems hold on each $[t_k, t_{k+1}]$. For $k = 0, 1, 2, \ldots$, let $f_k : [t_k, t_{k+1}] \times E \longrightarrow E$ where $f_k(t, x_k(t)) = f_k(t, x_k(t), \lambda_k(x_k))$. The HFDE (3) can be written in expanded form as:

$$x'(t) = \begin{cases} x_0'(t) = f(t, x_0(t), \lambda_0(x_0)) \equiv f_0(t, x_0(t)), \; x_0(t_0) = x_0, \; t_0 \in \Omega_0, \\ x_1'(t) = f(t, x_1(t), \lambda_1(x_1)) \equiv f_1(t, x_1(t)), \; x_1(t_1) = x_1, \; t_1 \in \Omega_1, \\ \vdots \\ x_k'(t) = f(t, x_k(t), \lambda_k(x_k)) \equiv f_k(t, x_k(t)), \; x_k(t_k) = x_k, \; t_k \in \Omega_k, \\ \vdots \end{cases} \tag{4}$$

where $\Omega_k = [t_k, t_{k+1}], \; k = 0, 1, 2, \ldots$. Using the solution of equation (4), we can obtain the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), & t_0 \in \Omega_0, \\ x_1(t), & t_1 \in \Omega_1, \\ \vdots \\ x_k(t), & t_k \in \Omega_k, \\ \vdots \end{cases} \tag{5}$$

A solution $x : [t_0, \infty) \longrightarrow E$ of equation (3) will be continuous and piecewise differentiable over $[t_0, \infty)$ and differentiable in each interval Ω_k for $k = 0, 1, 2, \ldots$. The following theorem can be proven from Theorem 2. over each Ω_k.

Theorem 3.1. [25] *Consider HFDE (3) expanded as (4) where for* $k = 0, 1, 2, \ldots$, *each* $f_k : \Omega_k \times E \longrightarrow E$, *is such that*

- $[f_k(t,x)]^r = [\underline{f}_k^r(t,\underline{x},\overline{x}), \overline{f}_k^r(t,\underline{x},\overline{x})]$,
- $\underline{f}_k^r$ *and* $\overline{f}_k^r$ *are equicontinuous and uniformly bounded on any bounded set,*
- *there exists an* $L_k > 0$ *such that*

$$|\underline{f}_k^r(t,x_1,y_1) - \underline{f}_k^r(t,x_2,y_2)| \leqslant L_k \max\{|x_2 - x_1|, |y_2 - y_1|\}, \quad r \in [0,1],$$

$$|\overline{f}_k^r(t,x_1,y_1) - \overline{f}_k^r(t,x_2,y_2)| \leqslant L_k \max\{|x_2 - x_1|, |y_2 - y_1|\}, \quad r \in [0,1].$$

Then, the hybrid fuzzy initial value problem (3) and system of ODEs

$$\begin{cases} (\underline{x}_k^r(t))' = \underline{f}_k^r(t, \underline{x}_k^r, \overline{x}_k^r), \\ (\overline{x}_k^r(t))' = \overline{f}_k^r(t, \underline{x}_k^r, \overline{x}_k^r), \\ \underline{x}_0^r(t_0) = \underline{x}_0^r, \ \underline{x}_k^r(t_k) = \underline{x}_{k-1}^r(t_k) \ if \ k > 0, \\ \overline{x}_0^r(t_0) = \overline{x}_0^r, \ \overline{x}_k^r(t_k) = \overline{x}_{k-1}^r(t_k) \ if \ k > 0, \end{cases} \tag{6}$$

are equivalent.

Kaleva [14] shows a useful procedure to solve the fuzzy differential equation. Consider the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0, \tag{7}$$

where $f : I \times E \to E$ is a continuous fuzzy mapping and x_0 is a fuzzy number. In the following we discuss the existence and uniqueness of the solutions using the method in [14].

Theorem 3.2. *Let* $f : I \times E \to E$ *be a continuous fuzzy mapping. There exists a* $L > 0$, *such that* $D(f(t,x), f(t,z)) \leqslant LD(x,z)$ *for all* $t \in I$ *and* $x, z \in E$. *Then problem (7) has unique solution on* I.

In HFDE (4), a fuzzy initial value problem (7) is created on each Ω_k, then by the above theorem the existence and uniqueness of solutions of the hybrid systems hold on each Ω_k.

4. The Homotopy Perturbation Method

To describe the basic ideas of the HPM, we consider the following nonlinear differential equation [11]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{8}$$

with the boundary conditions

$$B\Big(u, \frac{\partial u}{\partial n}\Big) = 0, \quad r \in \Omega, \tag{9}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. Generally, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Therefore, equation (8) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{10}$$

By the HPM, we construct a homotopy $v(r, p) : \Omega \times [0, 1] \longrightarrow \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \ r \in \Omega, \tag{11}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{12}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of equation (8). Obviously, from these definitions we will have

$$H(v, 0) = L(v) - L(u_0) = 0, \quad H(v, 1) = A(v) - f(r) = 0.$$

The process of changing of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopies. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of (12) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \dots. \tag{13}$$

Setting $p = 1$ results in the approximate solution of equation (13):

$$u = \lim_{p \longrightarrow 1} v = v_0 + v_1 + v_2 + \dots. \tag{14}$$

The series (14) is convergent for most cases, however, the convergent rate depends upon the nonlinear operator $A(v)$ [11]. Some conditions of convergence have been stated as follows:

- The second derivative of $N(v)$ with respect to v must be small, because the parameter p may be relatively large, i.e. $p \longrightarrow 1$.
- The norm of $L^{-1}\left(\frac{\partial N}{\partial v}\right)$ must be smaller than one, in order that the series converges.

The modified form of the HPM [22] can be established based on the assumption that the function $f(r)$ can be divided into two parts, namely $f_0(r)$ and $f_1(r)$,

$$f(r) = f_0(r) + f_1(r).$$

According to assumption $f(r) = f_0(r) + f_1(r)$, we can construct the homotopy $v(r, p) : \Omega \times [0, 1] \longrightarrow \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f_1(r)] = f_0(r), \tag{15}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f_1(r)] = f_0(r). \tag{16}$$

The suggestion was that only the part f_0 be assigned to the zeroth component u_0, whereas the remaining part f_1 be combined with the component u_1. In order to solve (3) by means of modified HPM, we define linear and nonlinear operators as $L = \frac{d}{dt}$, $N = -x(t)$ and known analytic functions as $f_0 = m(t)\lambda_k(x_k(t))$ and $f_1 = 0$. According to (16), we can construct a convex homotopy such that

$$H_k(v, p) = v'(t) - u_0'(t) + pu_0'(t) - pv(t) = m(t)\lambda_k(x(t_k)), \quad t \in \Omega_k. \tag{17}$$

By using Theorems 2. and 3., we can write equations (13) and (17) as follows:

$$\begin{cases} \underline{v}(t;r) = \underline{v}_0(t;r) + p\underline{v}_1(t;r) + p^2\underline{v}_2(t;r) + \dots, \\ \\ \overline{v}(t;r) = \overline{v}_0(t;r) + p\overline{v}_1(t;r) + p^2\overline{v}_2(t;r) + \dots, \end{cases} \tag{18}$$

and

$$\begin{cases} \underline{H}_k(v,p) = \underline{v}'(t;r) - \underline{u}_0'(t;r) + p\underline{u}_0'(t;r) - p\underline{v}(t;r) = m(t)\underline{\lambda}_k(x_k(t;r)), \\ \\ \overline{H}_k(v,p) = \overline{v}'(t;r) - \overline{u}_0'(t;r) + p\overline{u}_0'(t;r) - p\overline{v}(t;r) = m(t)\overline{\lambda}_k(x_k(t;r)), \end{cases} \tag{19}$$

for $t \in \Omega_k$. Substituting (18) in (19) and equating the coefficients of like powers of p, for $t \in \Omega_k$ yield

$$\begin{aligned} p^0 \quad &: \begin{cases} \underline{v}_0'(t;r) - \underline{u}_0'(t;r) = m(t)\underline{\lambda}_k(x_k(t;r)), \\ \\ \overline{v}_0'(t;r) - \overline{u}_0'(t;r) = m(t)\overline{\lambda}_k(x_k(t;r)), \end{cases} \\ p^1 \quad &: \begin{cases} \underline{v}_1'(t;r) + \underline{u}_0'(t;r) - \underline{v}_0(t;r) = 0, \\ \\ \overline{v}_1'(t;r) + \overline{u}_0'(t;r) - \overline{v}_0(t;r) = 0, \end{cases} \end{aligned} \tag{20}$$

$$\vdots$$

$$p^{n+1} : \begin{cases} \underline{v}_{n+1}'(t;r) - \underline{v}_n(t;r) = 0, \qquad n \geqslant 1, \\ \\ \overline{v}_{n+1}'(t;r) - \overline{v}_n(t;r) = 0, \qquad n \geqslant 1. \end{cases} \tag{21}$$

We start with initial approximation $\underline{u}_0(t_k;r) = \underline{x}(t_k)$ and $\overline{u}_0(t_k;r) = \overline{x}(t_k)$. For the sake of simplicity, we take $\underline{v}_0 = \underline{u}_0$ and $\overline{v}_0 = \overline{u}_0$. Since $\underline{v}_0(t_k;r) = \underline{u}_0(t_k;r)$, $\overline{v}_0(t_k;r) = \overline{u}_0(t_k;r)$ and $\underline{u} = \underline{v}_0 + \underline{v}_1 + \underline{v}_2 + \dots,\ \overline{u} = \overline{v}_0 + \overline{v}_1 + \overline{v}_2 + \dots$, we can set $\underline{v}_n(t_k;r) = \overline{v}_n(t_k;r) = 0$, $n = 1, 2, \dots$ as initial conditions for equations (20) and (21). So, we can get

$$p^0 : \begin{cases} \underline{v}_0(t;r) = \underline{v}_0(t_k;r) + \displaystyle\int_{t_k}^{t} \Big[\underline{u}_0'(s;r) + m(s)\underline{\lambda}_k(x_k(s;r))\Big] ds, \\ \\ \overline{v}_0(t;r) = \overline{v}_0(t_k;r) + \displaystyle\int_{t_k}^{t} \Big[\overline{u}_0'(s;r) + m(s)\overline{\lambda}_k(x_k(s;r))\Big] ds, \end{cases} \tag{22}$$

$$p^1 : \begin{cases} \underline{v}_1(t;r) = \displaystyle\int_{t_k}^{t} [\underline{v}_0(s;r) - \underline{u}_0'(s;r)]ds, \\ \\ \overline{v}_1(t;r) = \displaystyle\int_{t_k}^{t} [\overline{v}_0(s;r) - \overline{u}_0'(s;r)]ds, \end{cases} \tag{23}$$

$$\vdots$$

$$p^{n+1} : \begin{cases} \underline{v}_{n+1}(t;r) = \displaystyle\int_{t_k}^{t} \underline{v}_n(s;r)ds, \qquad n \geqslant 1, \\ \\ \overline{v}_{n+1}(t;r) = \displaystyle\int_{t_k}^{t} \overline{v}_n(s;r)ds, \qquad n \geqslant 1. \end{cases} \tag{24}$$

Then by (14), we obtain the approximate solution of equation (3) as:

$$\tilde{u}(t;r) = (\underline{\tilde{u}}(t;r), \tilde{\overline{u}}(t;r)), \tag{25}$$

where

$$\underline{\tilde{u}}(t;r) \cong \sum_{i=0}^{n} \underline{v}_i(t;r), \quad \tilde{\overline{u}}(t;r) \cong \sum_{i=0}^{n} \overline{v}_i(t;r). \tag{26}$$

and n is sufficiently large.

5. Convergence Analysis

In this section, the convergence of the HPM to solve HFDEs is studied according to the convergence of HPM discussed in [6].

In [34], it has been shown that fuzzy number space E can be embedded in to a concrete Banach space $\mathbf{B} = \bar{C}[0,1] \times \bar{C}[0,1]$, where the metric is usually defined as:

$$\|(u,v)\| = \max \left\{ \sup_{0 \leqslant r \leqslant 1} |u(r)|, \ \sup_{0 \leqslant r \leqslant 1} |v(r)| \right\},$$

for arbitrary $(u,v) \in \bar{C}[0,1] \times \bar{C}[0,1]$.

Consider equation (19) in the following form

$$\begin{cases} \underline{v}'(t;r) = m(t)\underline{\lambda}_k(x_k(t;r)) + (1-p)\underline{u}_0'(t;r) + p\underline{v}(t;r), & t \in \Omega_k, \\ \overline{v}'(t;r) = m(t)\overline{\lambda}_k(x_k(t;r)) + (1-p)\overline{u}_0'(t;r) + p\overline{v}(t;r), & t \in \Omega_k. \end{cases} \tag{27}$$

Applying the inverse operator, $\int_{t_k}^{t}(.)ds$ to both sides of equation (27), we obtain

$$\begin{cases} \underline{v}(t;r) = \underline{v}(t_k;r) + \displaystyle\int_{t_k}^{t}\Big[m(s)\underline{\lambda}_k(x_k(s;r)) + (1-p)\underline{u}_0'(s;r) + p\underline{v}(s;r))\Big]ds, \\ \overline{v}(t;r) = \overline{v}(t_k;r) + \displaystyle\int_{t_k}^{t}\Big[m(s)\overline{\lambda}_k(x_k(s;r)) + (1-p)\overline{u}_0'(s;r) + p\overline{v}(s;r)\Big]ds, \end{cases} \tag{28}$$

for $t \in \Omega_k$. Suppose that

$$\underline{v}(t;r) = \sum_{n=0}^{\infty} p^n \underline{v}_n(t;r), \quad \overline{v}(t;r) = \sum_{n=0}^{\infty} p^n \overline{v}_n(t;r), \tag{29}$$

substituting (29) into the right-hand side of equation (28), we have equation (28) in the following form for $t \in \Omega_k$,

$$\begin{cases} \underline{v}(t;r) = \displaystyle\sum_{n=0}^{\infty} p^n \underline{v}_n(t_k;r) \\ \qquad + \displaystyle\int_{t_k}^{t}\Big[m(s)\underline{\lambda}_k(x_k(s;r)) + (1-p)\underline{u}_0'(s;r) + p\sum_{n=0}^{\infty} p^n \underline{v}_n(s;r))\Big]ds, \\ \overline{v}(t;r) = \displaystyle\sum_{n=0}^{\infty} p^n \overline{v}_n(t_k;r) \\ \qquad + \displaystyle\int_{t_k}^{t}\Big[m(s)\overline{\lambda}_k(x_k(s;r)) + (1-p)\overline{u}_0'(s;r) + p\sum_{n=0}^{\infty} p^n \overline{v}_n(s;r)\Big]ds. \end{cases}$$

If $p \longrightarrow 1$, the exact solution may be obtained by using

$$\begin{cases} \underline{u}(t;r) = \lim\limits_{p \longrightarrow 1} \underline{v}(t;r) = \\ \qquad \displaystyle\sum_{n=0}^{\infty} \underline{v}_n(t_k;r) + \int_{t_k}^{t}\Big[m(s)\underline{\lambda}_k(x_k(s;r)) + \sum_{n=0}^{\infty} \underline{v}_n(s;r))\Big]ds, \\ \overline{u}(t;r) = \lim\limits_{p \longrightarrow 1} \overline{v}(t;r) = \\ \qquad \displaystyle\sum_{n=0}^{\infty} \overline{v}_n(t_k;r) + \int_{t_k}^{t}\Big[m(s)\overline{\lambda}_k(x_k(s;r)) + \sum_{n=0}^{\infty} \overline{v}_n(s;r)\Big]ds. \end{cases} \tag{30}$$

In this manner, we give the sufficient conditions for convergence of HPM, in following theorem.

Theorem 5.1. *Suppose that* $N : \mathbf{B} \longrightarrow \mathbf{B}$ *be a contractive nonlinear mapping, such that for every*

$$W,\ W^* \in \mathbf{B},\quad \|N(W) - N(W^*)\| \leqslant \gamma\ \|W - W^*\|,\quad 0 < \gamma < 1,$$

where $W = (\underline{W}, \overline{W})$ *and* $W^* = (\underline{W}^*, \overline{W}^*)$. *Then according to Banach's fixed point theorem* N *has a unique fixed point* $u = (\underline{u}, \overline{u}) \in \mathbf{B}$, *that is* $N(u) = u$. *Assume that the sequence generated by HPM can be written as*

$$W_n = (\underline{W}_n, \overline{W}_n) = N(W_{n-1}),\quad W_{n-1} = \left(\sum_{i=0}^{n-1} \underline{v}_i,\ \sum_{i=0}^{n-1} \overline{v}_i\right),\quad n \geqslant 1,$$

and suppose that $W_0 = (\underline{v}_0, \overline{v}_0) \in B_r(u) = \{u^* \in \mathbf{B} : \|u - u^*\| < r\}$, *then we have*

$$W_n \in B_r(u),\qquad \lim_{n \longrightarrow \infty} W_n = u.$$

Proof. By inductive approach, for $n = 1$ we have

$$\|\underline{W}_1 - \underline{u}\| = \|N(\underline{W}_0) - N(\underline{u})\| \leqslant \gamma\ \|\underline{W}_0 - \underline{u}\|.$$

Assume that $\|\underline{W}_{n-1} - \underline{u}\| \leqslant \gamma^{n-1}\ \|\underline{W}_0 - \underline{u}\|$, az induction hypothesis, then

$$\|\underline{W}_n - \underline{u}\| = \|N(\underline{W}_{n-1}) - N(\underline{u})\| \leqslant \gamma\ \|\underline{W}_{n-1} - \underline{u}\| \leqslant \gamma^n\ \|\underline{W}_0 - \underline{u}\|.$$

Since $W_0 \in B_r(u)$ and $\|\underline{W}_0 - \underline{u}\| < r$, then we have

$$\|\underline{W}_n - \underline{u}\| \leqslant \gamma^n\ \|\underline{W}_0 - \underline{u}\| \leqslant \gamma^n\ r < r,$$

Similarly $\|\overline{W}_n - \overline{u}\| \leqslant \gamma^n\ \|\overline{W}_0 - \overline{u}\| < r$, hence $W_n \in B_r(u)$. Because of

$$\|\underline{W}_n - \underline{u}\| \leqslant \gamma^n\ \|\underline{W}_0 - \underline{u}\|,\quad \|\overline{W}_n - \overline{u}\| \leqslant \gamma^n\ \|\overline{W}_0 - \overline{u}\|,$$

and $\lim_{n\longrightarrow\infty} \gamma^n = 0$ then we get

$$\lim_{n\longrightarrow\infty} \|\underline{W}_n - \underline{u}\| = 0, \quad \lim_{n\longrightarrow\infty} \|\overline{W}_n - \overline{u}\| = 0.$$

Then series solution $\left(\sum_{i=0}^{\infty} \underline{v}_i, \ \sum_{i=0}^{\infty} \overline{v}_i\right)$ are convergence to exact solution. $\square$

6. Numerical Results

In this section, we present three numerical examples with known analytical solution in order to show the efficiency and high accuracy of the HPM described for solving equation (3).

Example 6.1. [26] Consider the following hybrid fuzzy initial value problem

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), \ t \in \Omega_k, \ t_k = k, \ k = 1, 2, \ldots, \\ x(0; r) = (0.75 + 0.25r, \ 1.125 - 0.125r), \ 0 \leqslant r \leqslant 1, \end{cases} \tag{31}$$

where

$$m(t) = \begin{cases} 2(t(\text{mod } 1)), & \text{if } t(\text{mod } 1) \leqslant 0.5, \\ 2(1 - t(\text{mod } 1)), & \text{if } t(\text{mod } 1) \geqslant 0.5, \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0, \\ \mu, & \text{if } k \in \{1, 2, 3, \ldots\}, \end{cases}$$

for which $\hat{0} \in E$ is defined as:

$$\hat{0} = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Then $f(t, x(t), \lambda_k(x(t_k))) = x(t) + m(t)\lambda_k(x(t_k))$ is a continuous function of t, x and $\lambda_k(x(t_k))$. By Example 6.1 of Kaleva [15], for each $k = 0, 1, 2, \ldots,$ the hybrid fuzzy initial value problem

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)),\ t \in \Omega_k,\ t_k = k, \\ x(t_k) = x_k, \end{cases}$$

has a unique solution on Ω_k.
The exact solution for $t \in [0, 2]$ is given by

$$x(t;r) = \begin{cases} \Big(e^t(0.75 + 0.25r),\ e^t(\ 1.125 - 0.125r)\Big), & t \in [0,1], \\ x(1;r)\big[3e^{t-1} - 2t\big], & t \in [1, 1.5], \\ x(1;r)\Big[2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)\Big], & t \in [1.5, 2]. \end{cases}$$

To numerically solve equation (31), we will apply the HPM for HFDEs to obtain approximate solution $\tilde{x}(t;r) = (\underline{\tilde{x}}(t;r), \overline{\tilde{x}}(t;r))$, for $0 \leqslant r \leqslant 1$ and $t \in [0, 2]$.
For $k = 0$, the hybrid fuzzy initial value problem (31) becomes

$$\begin{cases} x'(t) = x(t),\ t \in [0,1], \\ x(0;r) = (0.75 + 0.25r,\ 1.125 - 0.125r),\ 0 \leqslant r \leqslant 1. \end{cases} \tag{32}$$

By Theorem 3.1, equation (32) is equivalent to the following system of ODEs

$$\begin{cases} \underline{x}'(t;r) = \underline{x}(t;r),\ \underline{x}(0;r) = 0.75 + 0.25r, \\ \overline{x}'(t;r) = \overline{x}(t;r),\ \overline{x}(0;r) =\ 1.125 - 0.125r, \end{cases} \tag{33}$$

for $0 \leqslant r \leqslant 1$ and $t \in [0, 1]$. Applying the relations (22)-(24), we get

$$\begin{cases} \underline{v}_0(t;r) = 0.75 + 0.25r, \\ \overline{v}_0(t;r) = 1.125 - 0.125r, \end{cases}$$

and

$$\begin{cases} \underline{v}_{n+1}(t;r) = \int_0^t \underline{v}_n(s;r)ds, \quad n \geqslant 1, \\ \overline{v}_{n+1}(t;r) = \int_0^t \overline{v}_n(s;r)ds, \quad n \geqslant 1. \end{cases}$$

Then the approximate solution $\tilde{x}(t,r) = (\tilde{\underline{x}}(t,r), \tilde{\overline{x}}(t,r))$ can be obtained as follows:

$$\tilde{\underline{x}}(t;r) \cong \sum_{i=0}^{n} \underline{v}_i(t;r), \quad \tilde{\overline{x}}(t;r) \cong \sum_{i=0}^{n} \overline{v}_i(t;r).$$

The absolute error functions of Example 6.1 with $n = 12$ and $k = 0$ for $(t,r) \in [0,1] \times [0,1]$ are plotted in Figs. 1 and 2, where $|\underline{x}(t;r) - \tilde{\underline{x}}(t;r)|$ and $|\overline{x}(t;r) - \tilde{\overline{x}}(t;r)|$ are lower and upper absolute error functions, respectively.

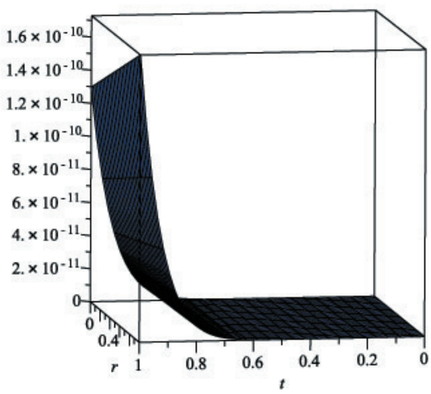

Figure 1: The lower absolute errors for Example 6.1 with $k = 0$ and $n = 12$.

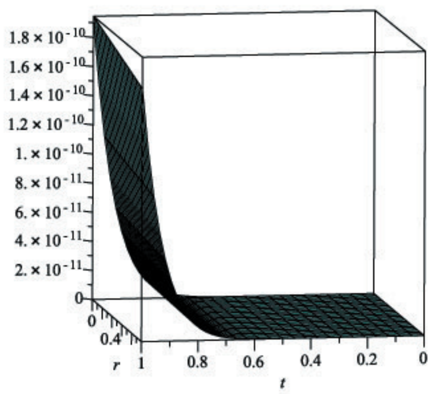

Figure 2: The upper absolute errors for Example 6.1 with $k = 0$ and $n = 12$.

For $k = 1$, the hybrid fuzzy initial value problem (31) can be written as:

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_1(x(t1)), \qquad t \in [1,2], \\ x(1;r) = \left(e^1(0.75 + 0.25r),\ e^1(1.125 - 0.125r)\right),\ 0 \leqslant r \leqslant 1. \end{cases} \tag{34}$$

Equation (34) is equivalent to the system of ODEs

$$\begin{cases} \underline{x}'(t;r) = \underline{x}(t;r) + m(t)\underline{x}(1;r),\ \underline{x}(1;r) = e^1(0.75 + 0.25r), \\ \overline{x}'(t;r) = \overline{x}(t;r) + m(t)\overline{x}(1;r),\ \overline{x}(1;r) = e^1(1.125 - 0.125r), \end{cases} \tag{35}$$

for which $0 \leqslant r \leqslant 1$, $t \in [1,2]$ and

$$m(t) = \begin{cases} 2t - 2, & 1 \leqslant t \leqslant 1.5, \\ 4 - 2t, & 1.5 \leqslant t \leqslant 2. \end{cases}$$

Using HPM, we obtain,

$$\begin{cases} \underline{v}_0(t;r) = e^1(0.75 + 0.25r)\left(1 + \int_1^t m(s)ds\right), \\ \overline{v}_0(t;r) = e^1(1.125 - 0.125r)\left(1 + \int_1^t m(s)ds\right), \end{cases}$$

and

$$\begin{cases} \underline{v}_{n+1}(t;r) = \int_1^t \underline{v}_n(s;r)ds, \quad n \geqslant 1, \\ \overline{v}_{n+1}(t;r) = \int_1^t \overline{v}_n(s;r)ds, \quad n \geqslant 1. \end{cases}$$

Then we have

$$\tilde{\underline{x}}(t;r) \cong \sum_{i=0}^{n} \underline{v}_i(t;r), \quad \tilde{\overline{x}}(t;r) \cong \sum_{i=0}^{n} \overline{v}_i(t;r).$$

The absolute error functions for Example 6.1 with $n = 10$ and $k = 1$ for $t \in [1,2]$ and $r \in [0,1]$ are plotted in Figs. 3 and 4.

Comparison of the approximate solution $\tilde{x}(t,r) = (\underline{\tilde{x}}(t;r), \overline{\tilde{x}}(t,r))$ and the exact solution $x(t,r) = (\underline{x}(t,r), \overline{x}(t,r))$, for $t = 1$ and $t = 2$ with $n = 12$ are given in Figs. 5 and 6.

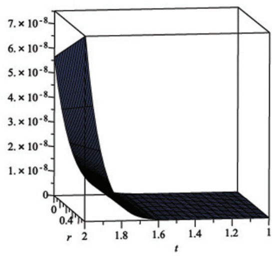

Figure 3: The lower absolute errors for Example 6.1 with $k = 1$ and $n = 10$.

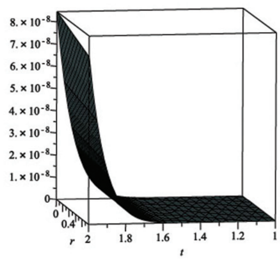

Figure 4: The upper absolute errors for Example 6.1. with $k = 1$ and $n = 10$.

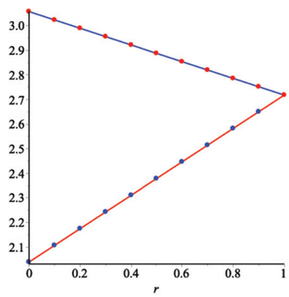

Figure 5: Comparison of exact the solution(solid graph) and the approximate solution(diamond curve) for Example 6.1 with $k = 0$ and $n = 12$ for $t = 1$.

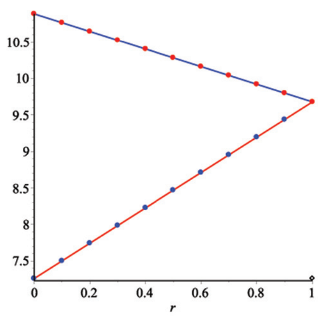

Figure 6: Comparison of exact the solution(solid graph) and the approximate solution(diamond curve) for Example 6.1 with $k = 1$ and $n = 12$ for $t = 2$.

The absolute errors obtained for presented method for Example 6.1 with $k = 0$, $k = 1$ and various n, are shown in Tables 1 and 2. The approximate solution $(\underline{\tilde{x}}, \tilde{\overline{x}})$ and the absolute error for Example 6.1 are demonstrated in Table 3 with $n = 14$ and $k = 0$ for $t = 0.5$ and in Table 4 with $n = 14$ and $k = 1$ for $t = 1.5$.
Let $(\underline{\tilde{x}}_{RRK4}, \tilde{\overline{x}}_{RRK4})$, $(\underline{\tilde{x}}_{RK4}, \tilde{\overline{x}}_{RK4})$ and $(\underline{\tilde{x}}_{Eu}, \tilde{\overline{x}}_{Eu.})$ be approximate solutions obtained from fourth-order fuzzy reduced Runge-Kutta method (RRK4) [1], Runge-Kutta method [27] and Euler method [26], respectivly. The comparison between absolute errors obtained from HPM and methods mentioned above, are shown in Tables 5 and 6.

Table 1: The absolute error for presented method for Example 6.1 with $k = 0$.

n	$\lvert\underline{x}(t,r) - \underline{\tilde{x}}(t,r)\rvert$	$\lvert\overline{x}(t,r) - \tilde{\overline{x}}(t,r)\rvert$
4	1.0×10^{-2}	1.1×10^{-2}
6	2.3×10^{-4}	2.5×10^{-4}
8	3.0×10^{-6}	3.5×10^{-6}
10	2.7×10^{-8}	3.1×10^{-8}
12	1.7×10^{-10}	1.9×10^{-10}
14	8.0×10^{-13}	9.0×10^{-13}

Table 2: The absolute error for presented method for Example 6.1 with $k = 1$.

n	$\lvert\underline{x}(t,r) - \underline{\tilde{x}}(t,r)\rvert$	$\lvert\overline{x}(t,r) - \tilde{\overline{x}}(t,r)\rvert$
4	2.8×10^{-2}	3.1×10^{-1}
6	6.0×10^{-4}	7.0×10^{-4}
8	8.5×10^{-6}	9.5×10^{-6}
10	7.5×10^{-8}	8.5×10^{-8}
12	4.7×10^{-10}	5.3×10^{-10}
14	2.2×10^{-12}	2.5×10^{-12}

Table 3: The approximate solution $(\underset{\sim}{x}, \tilde{\tilde{x}})$ and the absolute error for Example 6.1 with $k = 0$ and $n = 14$ for $t = 0.5$.

r	$\underset{\sim}{x}(0.5, r)$	*Absolute error*	$\tilde{\tilde{x}}(0.5, r)$	*Absolute error*
0	1.2365409530	1.81×10^{-17}	1.8548114295	2.71×10^{-17}
0.2	1.3189770165	1.93×10^{-17}	1.8135933976	2.65×10^{-17}
0.4	1.4014130800	2.05×10^{-17}	1.7723753660	2.59×10^{-17}
0.6	1.4838491436	2.17×10^{-17}	1.7311573342	2.53×10^{-17}
0.8	1.5662852071	2.29×10^{-17}	1.6899393024	2.47×10^{-17}
1	1.6487212707	2.41×10^{-17}	1.6487212707	2.41×10^{-17}

Table 4: The approximate solution $(\underset{\sim}{x}, \tilde{\tilde{x}})$ and the absolute error for Example 6.1 with $k = 1$ and $n = 14$ for $t = 1.5$.

r	$\underset{\sim}{x}(1.5, r)$	*Absolute error*	$\tilde{\tilde{x}}(1.5, r)$	*Absolute error*
0	3.9676662942	4.92×10^{-17}	5.9514994413	7.38×10^{-17}
0.2	4.2321773805	5.25×10^{-17}	5.8192438982	7.22×10^{-17}
0.4	4.4966884667	5.58×10^{-17}	5.6869883550	7.06×10^{-17}
0.6	4.7611995530	5.90×10^{-17}	5.5547328119	6.89×10^{-17}
0.8	5.0257106393	6.23×10^{-17}	5.4224772687	6.72×10^{-17}
1	5.2902217256	6.56×10^{-17}	5.2902217256	6.56×10^{-17}

Table 5: The comparison between absolute errors obtained from HPM with $k = 1$, $n = 12$ for $t = 2$ and other methods for Example 6.1.

r	$\lvert \underline{x} - \underset{\sim}{x}_{Eu.} \rvert$	$\lvert \underline{x} - \underset{\sim}{x}_{RRK4} \rvert$	$\lvert \underline{x} - \underset{\sim}{x}_{RK4} \rvert$	$\lvert \underline{x} - \underset{\sim}{x}_{HPM} \rvert$
0	4.03412×10^{-1}	1.08193×10^{-3}	1.73498×10^{-3}	3.55779×10^{-10}
0.2	4.30306×10^{-1}	1.15406×10^{-3}	1.85064×10^{-3}	3.79495×10^{-10}
0.4	4.57201×10^{-1}	1.22619×10^{-3}	1.96631×10^{-3}	4.03224×10^{-10}
0.6	4.84095×10^{-1}	1.29832×10^{-3}	2.08197×10^{-3}	4.26940×10^{-10}
0.8	5.10989×10^{-1}	1.37044×10^{-3}	2.19764×10^{-3}	4.50641×10^{-10}
1	5.37883×10^{-1}	1.44257×10^{-3}	2.31330×10^{-3}	4.74368×10^{-10}

Table 6: The comparison between absolute errors obtained from HPM with $k=1$, $n=12$ for $t=2$ and other methods for Example 6.1.

r	$\lvert\overline{x}-\tilde{\overline{x}}_{Eu.}\rvert$	$\lvert\overline{x}-\tilde{\overline{x}}_{RRK4}\rvert$	$\lvert\overline{x}-\tilde{\overline{x}}_{RK4}\rvert$	$\lvert\overline{x}-\tilde{\overline{x}}_{HPM}\rvert$
0	6.05119×10^{-1}	1.62290×10^{-3}	2.60247×10^{-3}	5.33680×10^{-10}
0.2	5.91672×10^{-1}	1.58683×10^{-3}	2.54464×10^{-3}	5.21820×10^{-10}
0.4	5.78224×10^{-1}	1.55077×10^{-3}	2.48680×10^{-3}	5.09960×10^{-10}
0.6	5.64777×10^{-1}	1.51470×10^{-3}	2.42897×10^{-3}	4.98110×10^{-10}
0.8	5.51330×10^{-1}	1.47864×10^{-3}	2.37114×10^{-3}	4.86237×10^{-10}
1	5.37883×10^{-1}	1.44257×10^{-3}	2.31330×10^{-3}	4.74368×10^{-10}

Example 6.2. [26] Consider the following HFDEs

$$\begin{cases} x'(t)=x(t)+m(t)\lambda_k(x(t_k)),\ t\in\Omega_k,\ t_k=k,\ k=1,2,...,\\[2ex] x(0;r)=(0.75+0.25r,\ 1.125-0.125r),\ 0\leqslant r\leqslant 1,\end{cases}\tag{36}$$

where

$$m(t)=|\sin(\pi t)|,\qquad \lambda_k(\mu)=\begin{cases}\hat{0},\quad \text{if } k=0,\\[2ex] \mu,\quad \text{if}\ \ k\in\{1,2,3,\ldots\}.\end{cases}$$

The exact solution for $t\in[0,2]$ satisfies

$$x(t;r)=\begin{cases}\left(e^t(0.75+0.25r),\ e^t(\ 1.125-0.125r)\right), & t\in[0,1],\\[2ex] x(1;r)\left(\frac{\sin(\pi t)+\pi cos(\pi t)}{\pi^2+1}+e^{t-1}(1+\frac{\pi}{\pi^2+1})\right), & t\in[1,2].\end{cases}$$

To numerically solve equation (36), we will apply the HPM for HFDEs to obtain the approximate solution $\tilde{x}(t;r)=(\underline{\tilde{x}}(t;r),\tilde{\overline{x}}(t;r))$, for $0\leqslant r\leqslant 1$ and $t\in[0,2]$.

For $k=0$, from Theorem 3.1, we conclude that equation (36) is equivalent to the system of ODEs (33) similar to Example 6.1 and thus the numerical solutions are the same.

Let $k=1$, then equation (36) is equivalent to the system of ODEs

$$\begin{cases}\underline{x}'(t;r)=\underline{x}(t;r)+m(t)\underline{x}(1;r),\ \underline{x}(1;r)=e^1(0.75+0.25r),\\[2ex] \overline{x}'(t;r)=\overline{x}(t;r)+m(t)\overline{x}(1;r),\ \overline{x}(1;r)=e^1(1.125-0.125r),\end{cases}$$

for which $0 \leqslant r \leqslant 1$, $t \in [1,2]$ and $m(t) = -\sin(\pi t)$. From HPM, we obtain,

$$\begin{cases} \underline{v}_0(t;r) = e^1(0.75+0.25r)\left(1+\int_1^t m(s)ds\right), \\ \overline{v}_0(t;r) = e^1(1.125-0.125r)\left(1+\int_1^t m(s)ds\right), \end{cases}$$

and

$$\begin{cases} \underline{v}_{n+1}(t;r) = \int_1^t \underline{v}_n(s;r)ds, \quad n \geqslant 1, \\ \overline{v}_{n+1}(t;r) = \int_1^t \overline{v}_n(s;r)ds, \quad n \geqslant 1. \end{cases}$$

Then we have

$$\underline{\tilde{x}}(t;r) \cong \sum_{i=0}^{n} \underline{v}_i(t;r), \quad \tilde{\overline{x}}(t;r) \cong \sum_{i=0}^{n} \overline{v}_i(t;r).$$

The absolute error functions for Example 6.2 with $k = 1$ and $n = 8$ are plotted in Figs. 7 and 8. The approximate solution $(\underline{\tilde{x}}, \tilde{\overline{x}})$ and the absolute error for Example 6.2 with $r = 0$ and $n = 10$ for $t \in [0,2]$ are shown in Table 7. The comparison between absolute errors obtained from HPM and methods mentioned before, are shown in Tables 8 and 9.

Table 7: The approximate solution $(\underline{\tilde{x}}, \tilde{\overline{x}})$ and the absolute error for Example 6.2 with $r = 0$ and $n = 10$ for $t \in [0,2]$.

t	$\underline{\tilde{x}}(t,0)$	*Absolute error*	$\tilde{\overline{x}}(t,0)$	*Absolute error*
0	0.7500000000	0.00	1.12500000000	0.00
0.2	0.9160520686	3.91×10^{-16}	1.37407810293	5.87×10^{-16}
0.4	1.1188685232	8.15×10^{-13}	1.67830278484	5.87×10^{-12}
0.6	0.9160520686	1.22×10^{-11}	1.37407810293	5.87×10^{-10}
0.8	1.6691556946	1.73×10^{-9}	2.50373354193	2.59×10^{-9}
1	2.0387113508	2.05×10^{-8}	3.05806702628	3.07×10^{-8}
1.2	2.2859753513	5.15×10^{-19}	3.42896302699	7.73×10^{-19}
1.4	3.5599761462	2.22×10^{-12}	5.33996421946	3.33×10^{-12}
1.6	4.7921428950	1.96×10^{-10}	7.18821434252	2.94×10^{-10}
1.8	6.2150716451	4.76×10^{-9}	9.32260746771	7.13×10^{-9}
2	7.7327506812	5.67×10^{-8}	11.5991260219	8.51×10^{-8}

Table 8: The comparison between absolute errors obtained from HPM with $k = 1$, $n = 12$ for $t = 2$ and other methods for Example 6.2.

r	$\lvert \underline{x} - \underline{\tilde{x}}_{Eu.} \rvert$	$\lvert \underline{x} - \underline{\tilde{x}}_{RRK4} \rvert$	$\lvert \underline{x} - \underline{\tilde{x}}_{RK4} \rvert$	$\lvert \underline{x} - \underline{\tilde{x}}_{HPM} \rvert$
0	2.65975×10^{-1}	8.61512×10^{-3}	8.40158×10^{-4}	3.57482×10^{-10}
0.2	2.83707×10^{-1}	9.18946×10^{-3}	8.96168×10^{-4}	3.81315×10^{-10}
0.4	3.01438×10^{-1}	9.76380×10^{-3}	9.52179×10^{-4}	4.05149×10^{-10}
0.6	3.19170×10^{-1}	1.03381×10^{-2}	1.00818×10^{-3}	4.28978×10^{-10}
0.8	3.36902×10^{-1}	1.09124×10^{-2}	1.06420×10^{-3}	4.52801×10^{-10}
1	3.54633×10^{-1}	1.14868×10^{-2}	1.12021×10^{-3}	4.76643×10^{-10}

Table 9: The comparison between absolute errors obtained from HPM with $k = 1$, $n = 12$ for $t = 2$ and other methods for Example 6.2.

r	$\lvert \overline{x} - \overline{\tilde{x}}_{Eu.} \rvert$	$\lvert \overline{x} - \overline{\tilde{x}}_{RRK4} \rvert$	$\lvert \overline{x} - \overline{\tilde{x}}_{RK4} \rvert$	$\lvert \overline{x} - \overline{\tilde{x}}_{HPM} \rvert$
0	3.98963×10^{-1}	1.29226×10^{-2}	1.54985×10^{-1}	5.36206×10^{-10}
0.2	3.90097×10^{-1}	1.26355×10^{-2}	1.24212×10^{-1}	5.24285×10^{-10}
0.4	3.81231×10^{-1}	1.23483×10^{-2}	9.34394×10^{-2}	5.12385×10^{-10}
0.6	3.72365×10^{-1}	1.20611×10^{-2}	6.26663×10^{-2}	5.00453×10^{-10}
0.8	3.63499×10^{-1}	1.17740×10^{-2}	3.18932×10^{-2}	4.88553×10^{-10}
1	3.54633×10^{-1}	1.14868×10^{-2}	1.12021×10^{-3}	4.74368×10^{-10}

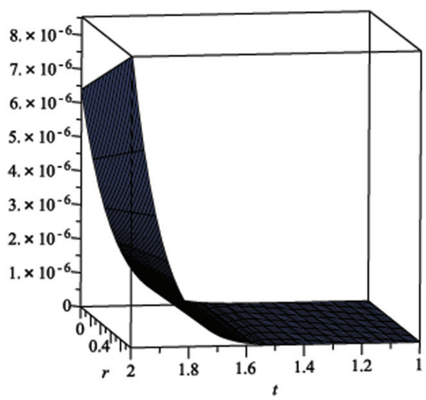

Figure 7: The lower absolute errors for Example 6.2 with $k = 1$ and $n = 8$

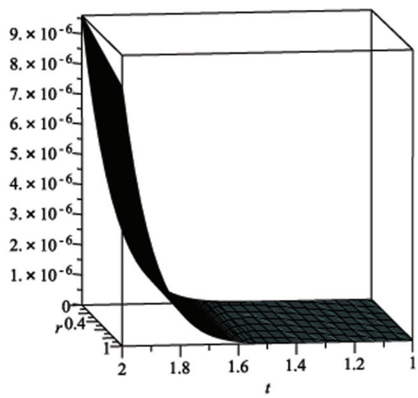

Figure 8: The upper absolute errors for Example 6.2 with $k = 1$ and $n = 8$.

Example 6.3 [28] Consider the following hybrid fuzzy initial value problem

$$\begin{cases} x'(t) = -x(t) + m(t)\lambda_k(x(t_k)),\ t \in \Omega_k,\ t_k = k,\ k = 1, 2, ..., \\ \\ x(0; r) = (0.75 + 0.25r,\ 1.125 - 0.125r),\ 0 \leqslant r \leqslant 1, \end{cases} \tag{37}$$

$$m(t) = |\sin(\pi t)|, \qquad \lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0, \\ \\ \mu, & \text{if } \ k \in \{1, 2, 3, \ldots\}. \end{cases}$$

The exact solution for $t \in [0, 1]$ satisfies

$$x(t; r) = \begin{cases} \underline{x}(t; r) = e^t(-0.1875 + 0.1875r) + e^{-t}(0.9375 + 0.0625r), \\ \\ \overline{x}(t; r) = e^t(0.1875 - 0.1875r) + e^{-t}(0.9375 + 0.0625r), \end{cases}$$

and for $t \in [1, 2]$ is given by $x(t; r) = x(1; r)(\underline{y}(t; r), \overline{y}(t, r))$ where

$$\begin{aligned} \underline{y}(t; r) = r &\left[e^{-t} - \frac{e^{-1}(\sin(\pi t) - \pi\cos(\pi t)) - \pi e^{-t}}{1 + (\pi)^2}\right] \\ &+(1 - r)\left[-0.1875\left(e^t + \frac{e^1(\sin(\pi t) + \pi\cos(\pi t)) + \pi e^t}{\pi^2 + 1}\right)\right] \\ &+(1 - r)\left[0.9375\left(e^{-t} - \frac{e^{-1}(\sin(\pi t) - \pi\cos(\pi t)) - \pi e^{-t}}{\pi^2 + 1}\right)\right], \end{aligned}$$

and

$$\begin{aligned} \overline{y}(t; r) = r &\left[e^{-t} - \frac{e^{-1}(\sin(\pi t) - \pi\cos(\pi t)) - \pi e^{-t}}{1 + (\pi)^2}\right] \\ &+(1 - r)\left[0.1875\left(e^t + \frac{e^1(\sin(\pi t) + \pi\cos(\pi t)) + \pi e^t}{\pi^2 + 1}\right)\right] \\ &+(1 - r)\left[0.9375\left(e^{-t} - \frac{e^{-1}(\sin(\pi t) - \pi\cos(\pi t)) - \pi e^{-t}}{\pi^2 + 1}\right)\right]. \end{aligned}$$

To numerically solve this example, similar to Examples 6. and 6., consider two steps as bellow:

Step 1): Let $k = 0$. From Theorem 3.1, equation (37) is equivalent to the system of ODEs

$$\begin{cases} \underline{x}'(t;r) = -\overline{x}(t;r), \ \underline{x}(0;r) = 0.75 + 0.25r, \\ \overline{x}'(t;r) = -\underline{x}(t;r), \ \overline{x}(0;r) = 1.125 - 0.125r, \end{cases} \tag{38}$$

for $0 \leqslant r \leqslant 1$ and $t \in [0,1]$. Using HPM, we get,

$$\begin{cases} \underline{v}_0(t;r) = 0.75 + 0.25r, \\ \overline{v}_0(t;r) = 1.125 - 0.125r, \end{cases}$$

and

$$\begin{cases} \underline{v}_{n+1}(t;r) = -\int_0^t \overline{v}_n(s;r)ds, \quad n \geqslant 1, \\ \overline{v}_{n+1}(t;r) = -\int_0^t \underline{v}_n(s;r)ds, \quad n \geqslant 1. \end{cases}$$

Then we have

$$\underline{\tilde{x}}(t;r) \cong \sum_{i=0}^{n} \underline{v}_i(t;r), \quad \tilde{\overline{x}}(t;r) \cong \sum_{i=0}^{n} \overline{v}_i(t;r).$$

Step 2): Let $k = 1$, then equation (37) is equivalent to the system of ODEs

$$\begin{cases} \underline{x}'(t;r) = -\overline{x}(t;r) + m(t)\underline{x}(1;r), \\ \overline{x}'(t;r) = -\underline{x}(t;r) + m(t)\overline{x}(1;r), \\ \underline{x}(1;r) = e^1(-0.1875 + 0.1875r) + e^{-1}(0.9375 + 0.0625r), \\ \overline{x}(1;r) = e^1(0.1875 - 0.1875r) + e^{-1}(0.9375 + 0.0625r). \end{cases}$$

for which $0 \leqslant r \leqslant 1$, $t \in [1,2]$ and $m(t) = -\sin(\pi t)$. Applying HPM, we obtain,

$$\begin{cases} \underline{v}_0(t;r) = \left[e^1(-0.1875 + 0.1875r) + e^{-1}(0.9375 + 0.0625r)\right](1 + \int_1^t m(s)ds), \\ \overline{v}_0(t;r) = \left[e^1(0.1875 - 0.1875r) + e^{-1}(0.9375 + 0.0625r)\right](1 + \int_1^t m(s)ds), \end{cases}$$

and

$$\begin{cases} \underline{v}_{n+1}(t;r) = -\int_1^t \overline{v}_n(s;r)ds, & n \geqslant 1, \\ \overline{v}_{n+1}(t;r) = -\int_1^t \underline{v}_n(s;r)ds, & n \geqslant 1. \end{cases}$$

Then we have

$$\underline{\tilde{x}}(t;r) \cong \sum_{i=0}^{n} \underline{v}_i(t;r), \quad \overline{\tilde{x}}(t;r) \cong \sum_{i=0}^{n} \overline{v}_i(t;r).$$

The absolute error functions for Example 6.1 with $k = 1$ and $n = 14$ are plotted in Figs. 9 and 10. Comparison of the approximate solution $\tilde{x}(t,r) = (\underline{\tilde{x}}(t;r), \overline{\tilde{x}}(t,r))$ and the exact solution $x(t,r) = (\underline{x}(t,r), \overline{x}(t,r))$, for $t = 1$ and $t = 2$ with $n = 14$ are given in Figs. 11 and 12.

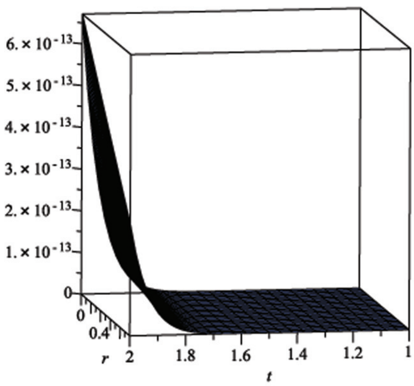

Figure 9: The lower absolute errors for Example 6.3 with $k = 1$ and $n = 14$

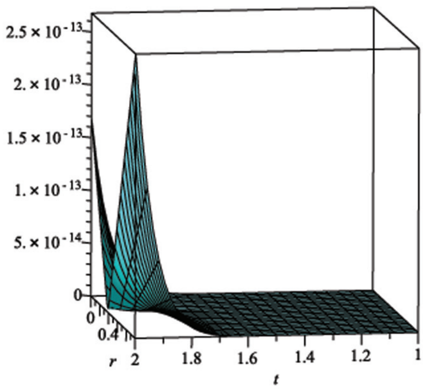

Figure 10: The upper absolute errors for Example 6.3 with $k = 1$ and $n = 14$

The absolute errors for presented method for Example 6.3 with $k = 0$, $k = 1$ and various n, are shown in Tables 10 and 11. The approximate solution $(\underline{\tilde{x}}, \overline{\tilde{x}})$ and the absolute error for Example 6. are shown in Table 6. with $n = 12$ and $k = 0$ for $t = 0.5$ and in Table 6. with $n = 12$ and $k = 1$ for $t = 1.5$. The approximate solution $(\underline{\tilde{x}}, \overline{\tilde{x}})$ and the absolute error for $r = 0$ and $t \in [0, 2]$ with $n = 10$ are demonstrated in Table 6..

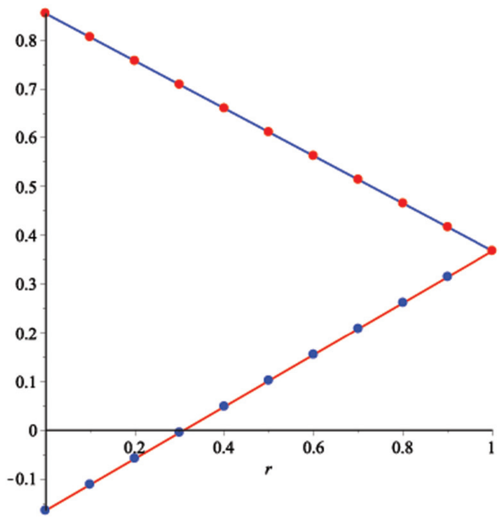

Figure 11: Comparison of the approximate solution (solid graph) and the exact solution (diamond curve) for Example 6.3 with $k = 0$ and $n = 14$ for $t = 1$

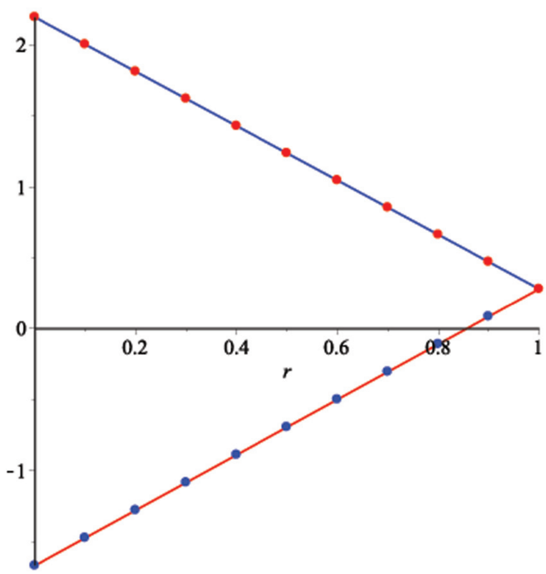

Figure 12: Comparison of the approximate solution (solid graph) and the exact solution (diamond curve) for Example 6.3 with $k = 1$ and $n = 14$ for $t = 2$

Table 10: The absolute error for presented method for Example 6. with $k = 0$.

n	$\lvert\underline{x}(t,r) - \tilde{\underline{x}}(t,r)\rvert$	$\lvert\overline{x}(t,r) - \tilde{\overline{x}}(t,r)\rvert$
4	8.6×10^{-3}	7.1×10^{-3}
6	2.1×10^{-4}	1.7×10^{-4}
8	2.9×10^{-6}	2.5×10^{-6}
10	2.7×10^{-8}	2.3×10^{-8}
12	1.6×10^{-10}	1.5×10^{-10}
14	8.3×10^{-13}	7.2×10^{-13}

Table 11: The absolute error for presented method for Example 6. with $k = 1$.

n	$\lvert\underline{x}(t,r) - \tilde{\underline{x}}(t,r)\rvert$	$\lvert\overline{x}(t,r) - \tilde{\overline{x}}(t,r)\rvert$
4	8.0×10^{-3}	2.8×10^{-3}
6	1.8×10^{-4}	6.6×10^{-5}
8	2.4×10^{-6}	9.3×10^{-7}
10	2.2×10^{-8}	8.6×10^{-9}
12	1.4×10^{-10}	5.6×10^{-11}
14	6.2×10^{-13}	2.6×10^{-13}

Table 12: The approximate solution $(\tilde{\underline{x}}, \tilde{\overline{x}})$ and the absolute error for Example 6. with $k = 0$ and $n = 12$ for $t = 0.5$.

r	$\tilde{\underline{x}}(0.5, r)$	*Absolute error*	$\tilde{\overline{x}}(0.5, r)$	*Absolute error*
0	0.2594872552	2.15×10^{-14}	0.8777577318	1.39×10^{-14}
0.2	0.3288959361	2.10×10^{-14}	0.8235123173	1.49×10^{-14}
0.4	0.3983046170	2.05×10^{-14}	0.7692669029	1.59×10^{-14}
0.6	0.4677132979	1.90×10^{-14}	0.7150214886	1.69×10^{-14}
0.8	0.5371219787	1.94×10^{-14}	0.6607760741	1.79×10^{-14}
1	0.6065306598	1.89×10^{-14}	0.6065306598	1.89×10^{-14}

Table 13: The approximate solution $(\underset{\sim}{x}, \tilde{\tilde{x}})$ and the absolute error for Example 6. with $k = 1$ and $n = 12$ for $t = 1.5$.

r	$\underset{\sim}{x}(1.5, r)$	*Absolute error*	$\tilde{\tilde{x}}(1.5, r)$	*Absolute error*
0	−0.734925776	1.69×10^{-14}	1.3376731186	3.85×10^{-15}
0.2	−0.523647571	1.49×10^{-14}	1.1344315447	7.44×10^{-14}
0.4	−0.312369368	1.30×10^{-14}	0.9311899696	4.87×10^{-16}
0.6	−0.101091160	1.10×10^{-14}	0.7279483967	2.65×10^{-15}
0.8	0.1101870436	8.98×10^{-15}	0.5247068225	4.82×10^{-15}
1	0.3214652488	6.99×10^{-15}	0.3214652486	6.99×10^{-15}

Table 14: The approximate solution $(\underset{\sim}{x}, \tilde{\tilde{x}})$ and the absolute error for Example 6. with $r = 0$ and $n = 10$ for $t \in [0, 2]$.

t	$\underset{\sim}{x}(t, 0)$	*Absolute error*	$\tilde{\tilde{x}}(t, 0)$	*Absolute error*
0	0.7500000000	0.00	1.1250000000	0.00
0.2	0.5385470638	1.47×10^{-19}	0.9965730982	9.66×10^{-19}
0.4	0.3487079124	1.19×10^{-15}	0.9081421740	7.74×10^{-15}
0.6	0.1728636338	2.30×10^{-13}	0.8561581840	1.47×10^{-13}
0.8	0.0039569797	9.58×10^{-12}	0.8385348279	6.07×10^{-12}
1	−0.1647908399	1.73×10^{-10}	0.8545648355	1.08×10^{-10}
1.2	−0.2542294175	1.37×10^{-23}	0.8887582564	2.74×10^{-24}
1.4	−0.5926177530	9.29×10^{-16}	1.1873703205	2.04×10^{-16}
1.6	−0.8930718791	1.82×10^{-13}	1.5029995663	4.25×10^{-14}
1.8	−1.2547167397	7.72×10^{-12}	1.8528190837	1.91×10^{-12}
2	−1.6699590953	1.42×10^{-10}	2.1964162684	3.69×10^{-11}

7. Conclusion

In this paper, an efficient method was presented to solve HFDEs by applying HPM. The sufficient conditions for convergence of this method for solving HFDEs were discussed in detail. In addition, several numerical examples of the proposed method were presented. The numerical results and graphical representations confirmed that our method is capable of generating more accurate solutions as compared to other methods such as fourth-order fuzzy reduced Runge-Kutta, Runge-Kutta and Euler in literature.

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