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Extension Principle Based on Neutrosophic Multi-Fuzzy Sets and Algebraic Operations

Mehmet Şahin
Gaziantep University

Irfan Deli
Kilis 7 Aralık University

Vakkas Uluçay*
Gaziantep University

Abstract. In this paper, we first proposed the extension principles of neutrosophic multi-sets and cut sets which are a bridge between neutrosophic multi-sets and crisp sets. Then the representation theorem of neutrosophic multi-sets based on cut sets are studied. Finally, the addition, subtraction, multiplication and division operations over neutrosophic multi-sets are defined based on the extension principle.

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1. Introduction

There are many mathematical tools that with uncertainties such as; rough set theory [19], fuzzy set theory [43], intuitionistic fuzzy set theory [7] and neutrosophic set theory [26]. A neutrosophic set theory independently is characterized by a truth-membership, a indeterminacy-membership and a falsity-membership. Therefore the neutrosophic set

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*Corresponding author

theory has become a popular topic of investigation in the real problems which contain uncertainty. In recent years, the academic community has witnessed growing research interests in neutrosophic set theory [1, 2, 3, 4, 5, 6, 8, 13, 15, 27, 38, 41].

Since membership values are inadequate for providing complete information in some real problems which has different membership values for each element, different generalization of fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets have been introduced is called multi fuzzy set [39], intuitionistic fuzzy multiset [36] and neutrosophic multiset [41, 42], respectively. In the multisets an element of a universe can be constructed more than once with possibly the same or different membership values. Some work on the multi fuzzy set [18, 28, 29, 30, 31, 32, 33, 34, 35], on intuitionistic fuzzy multiset [12, 16, 20, 21, 22, 23, 24, 25, 36] and on neutrosophic multiset [9, 10, 11, 37] have been studied.

As far as we know, however, there is less investigation on the cut sets and extension principle for neutrosophic set theory as well as algebraic operations. It is noted that the algebraic operations defined in this paper are remarkably different from those introduced by Chatterjee et al. [11], Deli et al [14] and Ye [42]. The purpose of this paper is to develop a bridge between neutrosophic multi-fuzzy sets and crisp sets. To do this, in Section 2, some concepts relate to neutrosophic sets and neutrosophic multi-fuzzy sets are briefly reviewed. In Section 3, the cut sets and extension principles of neutrosophic multi-fuzzy sets are developed. In Section 4, based on the extension principle some arithmetic operations such as; addition, subtraction, multiplication and division operations over neutrosophic multi-fuzzy sets are defined. In last Section, conclusion is made.

2. Definitions and Preliminary

In this section, we present the basic definitions and results of neutrosophic set theory [2, 38, 41] and neutrosophic multi (or refined) set theory [11, 14, 42] that are useful for subsequent discussions. See especially [2, 1, 9, 10, 11, 27, 37, 38, 41] for further details and background.

Definition 2.1. ([38]) *Let E be a space of points (objects), with a generic element in E denoted by u. A neutrosophic set (N-set) in U is characterized by a truth-membership function T_N, a indeterminacy-membership function I_N and a falsity-membership function F_N. $T_N(x)$, $I_N(x)$ and $F_N(x)$ are real standard or nonstandard subsets of $]^{-}0, 1^{+}[$.*
It can be written as

$$N = \{< x, (T_N(x), I_N(x), F_N(x)) >: x \in E, T_N(u), I_N(x), F_N(x) \subseteq [0, 1]\}.$$

There is no restriction on the sum of $T_N(u)$; $I_N(u)$ and $F_N(u)$, so $^{-}0 \leqslant supT_N(u) + supI_N(u) + supF_N(u) \leqslant 3^{+}$.
Here, 1^{+}= 1+ε, where 1 is its standard part and ε its non-standard part. Similarly, $^{-}0$= 1+ε, where 0 is its standard part and ε its non-standard part.

Definition 2.2. [2] *Let N and M be two neutrosophic set on E. Then,*

1. *A neutrosophic set N with $T_N(u) = 1$, $I_N(u) = 1$,$F_N(u) = 1$ is called normal neutrosophic set.*
2. *When the support set is a real number set and the following applies for all $u \in [a, b]$ over any interval $[a, b]$; $T_N(u) \geqslant T_M(a) \wedge T_M(b)$, $I_N(u) \geqslant I_M(a) \wedge I_M(b)$ and $F_N(u) \geqslant F_M(a) \wedge F_M(b)$*

   A is said to be convex.
3. *For a neutrosophic set $N = \{< u, (T_N(u), I_N(u), F_N(u)) >: u \in E\}$*

   $N_\alpha = \{u : u \in U, either T_N(u), I_N(u) > \alpha \, or \, F_N(u) < 1 - \alpha, \alpha \in]^{-}0, 1^{+}[\}$ *and*

   $N_{\overline{\alpha}} = \{u : u \in U, either T_N(u), I_N(u) \geqslant \alpha \, or \, F_N(u) \leqslant 1 - \alpha, \alpha \in]^{-}0, 1^{+}[\}$ *are called the weak and strong cut respectively.*
4. *Extending the function $f : X \to Y$, the neutrosophic subset N of X is made to correspond to neutrosophic subset $f(N) = \{T_{f(N)}, I_{f(N)}, F_{f(N)}\}$ of Y may be the following ways*

$$T_{f(N)}(y) = \begin{cases} \vee\{T_N(x) : x \in f^{-1}(y)\}, & iff^{-1}(y) \neq \emptyset, \\ 0, & otherwise \end{cases}$$

$$I_{f(N)}(y) = \begin{cases} \wedge\{I_N(x) : x \in f^{-1}(y)\}, & iff^{-1}(y) \neq \emptyset, \\ 0, & otherwise \end{cases}$$

$$F_{f(N)}(y) = \begin{cases} \wedge\{F_N(x) : x \in f^{-1}(y)\}, & iff^{-1}(y) \neq \emptyset, \\ 0, & otherwise. \end{cases}$$

3. Extension Principles and Cut Sets for Neutrosophic Multi-Sets

In this section, we present the extension principles of neutrosophic multi-sets and the representation theorem of neutrosophic multi-sets based on cut sets by extending the study are given in [2, 17].

Definition 3.1. *Let E be a universe and $N_p = \{1, 2, ...p\}$. A neutrosophic multi-fuzzy set(Nmfs) A on E is the set*
$A = \{\langle x, (T_A^1(x), T_A^2(x), ..., T_A^p(x)), (I_A^1(x), I_A^2(x), ..., I_A^p(x)),$
$(F_A^1(x), F_A^2(x), ..., F_A^p(x))\rangle : x \in E\}$, *where* $T_A^i(x), I_A^j(x), F_A^k(x) \in [0, 1]$ *with* $0 \leqslant supT_A^i(x) + supI_A^j(x) + supF_A^k(x) \leqslant 3$ *for* $i, j, k \in N_p$.
$(T_A^1(x), T_A^2(x), ..., T_A^P(x)), (I_A^1(x), I_A^2(x), ..., I_A^P(x))$ *and* $(F_A^1(x), F_A^2(x), ..., F_A^P(x))$ *are the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively. Also, p is called the dimension (cardinality) of Nmfs A and it is denoted by $d(A)$.*

Definition 3.2. *If a neutrosophic multi-sets A satisfies the following conditions, we call it a neutrosophic multi-numbers.*

1. *A neutrosophic multi-sets A with $T_A^1(u) = T_A^2(u) = ... = T_A^P(u) = 1$, $I_A^1(u) = I_A^2(u) = ... = I_A^P(u) = 0$, $F_A^1(u) = F_A^2(u) = ... = F_A^P(u) = 0$ is called normal neutrosophic multi-sets.*

2. *When the support set is a real number set and the following applies for all $u \in [a, b]$ over any interval $[a, b]$; $T_A^i(u) \geqslant T_B^i(a) \wedge T_B^i(b)$, $I_A^i(u) \leqslant I_B^i(a) \vee I_B^i(b)$ and $F_A^i(u) \leqslant F_B^i(a) \vee F_B^i(b)$*

   A is said to be convex (T_A^i; convex, I_A^i and F_A^i; concave), ($i \in \{1, 2, ..., p\}$).

Definition 3.3. *Let* $A = \{< u, (T_A^1(u), T_A^2(u), ..., T_A^P(u)), (I_A^1(u), I_A^2(u), ..., I_A^P(u)), (F_A^1(u), F_A^2(u), ..., F_A^P(u)) >: u \in U\}, \{T_A^i(u), I_A^i(u), F_A^i(u) \in [0,1]\}, (i \in \{1, 2, ..., p\})$ *be any neutrosophic multi-sets on U. Then, extending the function* $f : X \to Y$*, the neutrosophic multi-subsets A of X is made to correspond to neutrosophic multi-subsets* $f(A) = \{T_{f(A)}^i, I_{f(A)}^i, F_{f(A)}^i\}$ *of Y may be the following ways*

$$T_{f(A)}^i(y) = \begin{cases} \vee\{T_A^i(x) : x \in f^{-1}(y)\}, & if f^{-1}(y) \neq \emptyset, \\ 0, & otherwise \end{cases}$$

$$I_{f(A)}^i(y) = \begin{cases} \wedge\{I_A^i(x) : x \in f^{-1}(y)\}, & if f^{-1}(y) \neq \emptyset, \\ 1, & otherwise \end{cases}$$

$$F_{f(A)}^i(y) = \begin{cases} \wedge\{F_A^i(x) : x \in f^{-1}(y)\}, & if f^{-1}(y) \neq \emptyset, \\ 1, & otherwise \end{cases}$$

for $i = 1, 2, ...p$.

Definition 3.4. *Let* $A = \{< u, (T_A^1(u), T_A^2(u), ..., T_A^P(u)), (I_A^1(u), I_A^2(u), ..., I_A^P(u)), (F_A^1(u), F_A^2(u), ..., F_A^P(u)) >: u \in U\}, \{T_A^i(u), I_A^i(u), F_A^i(u) \in [0,1]\}, (i \in \{1, 2, ..., p\})$ *be any neutrosophic multi-sets on U. For any ordered a* (α, β, γ) *where* $\alpha, \beta, \gamma \in [0, 1]$*, so* $0 \leqslant \alpha + \beta + \gamma \leqslant 3$*. Then* (α, β, γ)*-cut set of neutrosophic multi-sets A is denoted by* $A_{(\alpha,\beta,\gamma)}$*, is defined by*

$$A_{(\alpha,\beta,\gamma)} = \{u : T_A^i(u) \geqslant \alpha, I_A^i(u) \leqslant \beta, F_A^i(u) \leqslant \gamma, u \in U\},$$

That is;

$$A_{(\alpha,\beta,\gamma)} = \{u : T_A^i(u) \wedge \alpha = \alpha, I_A^i(u) \vee \beta = \beta, F_A^i(u) \vee \gamma = \gamma, u \in U\}.$$

The strong (α, β, γ)*-cut set of neutrosophic multi-sets A is denoted by* $A_{\overline{(\alpha,\beta,\gamma)}}$*, is defined by*

$$A_{\overline{(\alpha,\beta,\gamma)}} = \{u : T_A^i(u) > \alpha, I_A^i(u) < \beta, F_A^i(u) < \gamma, u \in U\},$$

Definition 3.5. *Let* $A = \{< u, (T_A^1(u), T_A^2(u), ..., T_A^P(u)), (I_A^1(u), I_A^2(u),$

..., $I_A^P(u)), (F_A^1(u), F_A^2(u), ..., F_A^P(u)) >: u \in U\}, \{T_A^i(u), I_A^i(u), F_A^i(u) \in [0,1]\}, (i \in \{1,2,...,p\})$ be any neutrosophic multi-sets on U and $A_{(\alpha,\beta,\gamma)}$ be any (α,β,γ)-cut set of neutrosophic multi-sets A. Then,

1. *$A_{((1,1,...,1),(0,0,...0),(0,0,...,0))}$ is called the core of neutrosophic multi-sets A.*
2. *$A_{((0,0,...,0),(1,1,...1),(1,1,...,1))}$ is called the support of neutrosophic multi-sets A.*

Remark 3.6. *For any ordered (α,β,γ), a crisp set on X can always be derived from the neutrosophic multi-sets A. Its an inverse image of the neutrosophic multi-sets A at the confidence cut (α,β,γ).*

Theorem 3.7. *Let $A = \{< u, (T_A^1(u), T_A^2(u), ..., T_A^P(u)), (I_A^1(u), I_A^2(u), ..., I_A^P(u)), (F_A^1(u), F_A^2(u), ..., F_A^P(u)) >: u \in U\}, \{T_A^i(u), I_A^i(u), F_A^i(u) \in [0,1]\}, (i \in \{1,2,...,p\})$ be any neutrosophic multi-sets on U. Then, $(\alpha_1,\beta_1,\gamma_1) \leqslant (\alpha_2,\beta_2,\gamma_2)$*

1. *$A_{(\alpha_2,\beta_2,\gamma_2)} \subseteq A_{(\alpha_1,\beta_1,\gamma_1)}$.*
2. *$A_{\overline{(\alpha_2,\beta_2,\gamma_2)}} \subseteq A_{\overline{(\alpha_1,\beta_1,\gamma_1)}}$.*
3. *$A_{(\alpha_2,\beta_2,\gamma_2)} \subseteq A_{\overline{(\alpha_1,\beta_1,\gamma_1)}}$.*

Here, $(\alpha_1,\beta_1,\gamma_1) \leqslant (\alpha_2,\beta_2,\gamma_2)$ means that $\alpha_1 \leqslant \alpha_2, \beta_1 \geqslant \beta_2$ and $\gamma_1 \geqslant \gamma_2$

Proof. Here, $(\alpha_1,\beta_1,\gamma_1) \leqslant (\alpha_2,\beta_2,\gamma_2)$ means that $\alpha_1 \leqslant \alpha_2, \beta_1 \geqslant \beta_2$ *and* $\gamma_1 \geqslant \gamma_2$

$$\begin{aligned} A_{(\alpha_2,\beta_2,\gamma_2)} &= \{u : T_A^i(u) \geqslant \alpha_2, I_A^i(u) \leqslant \beta_2, F_A^i(u) \leqslant \gamma_2, u \in U\} \\ &\leqslant \{u : T_A^i(u) \geqslant \alpha_1, I_A^i(u) \leqslant \beta_1, F_A^i(u) \leqslant \gamma_1, u \in U\} \\ &\subseteq A_{(\alpha_1,\beta_1,\gamma_1)}. \end{aligned}$$

Therefore $A_{(\alpha_2,\beta_2,\gamma_2)} \subseteq A_{(\alpha_1,\beta_1,\gamma_1)}$.
2.

$$\begin{aligned} A_{\overline{(\alpha_2,\beta_2,\gamma_2)}} &= \{u : T_A^i(u) > \alpha_2, I_A^i(u) < \beta_2, F_A^i(u) < \gamma_2, u \in U\} \\ &< \{u : T_A^i(u) > \alpha_1, I_A^i(u) < \beta_1, F_A^i(u) < \gamma_1, u \in U\} \\ &\subseteq A_{\overline{(\alpha_1,\beta_1,\gamma_1)}}. \end{aligned}$$

Therefore $A_{\overline{(\alpha_2,\beta_2,\gamma_2)}} \subseteq A_{\overline{(\alpha_1,\beta_1,\gamma_1)}}.$
3. It is proved similarly. $\square$

Example 3.8. Suppose that $A = \{u_1 : (0.6, 0.5, 0.4, 0.3), (0.3, 0.2, 0.4, 0.5), (0.4, 0.6, 0.8, 0.7), u_2 : (0.5, 0.8, 0.3, 0.4), (0.6, 0.4, 0.3, 0.5), (0.3, 0.5, 0.4, 0.6), u_3 : (0.7, 0.2, 0.4, 0.5), (0.2, 0.6, 0.1, 0.4), (0.1, 0.3, 0.7, 0.5)\}$. Let $(\alpha_1 = 0.5, \beta_1 = 0.4, \gamma_1 = 0.3) \in [0, 1]$ and $(\alpha_2 = 0.7, \beta_2 = 0.1, \gamma_2 = 0.2) \in [0, 1]$.Then
1.$A_{(\alpha_1,\beta_1,\gamma_1)} = \{u_1, u_2, u_3\}$ and $A_{(\alpha_2,\beta_2,\gamma_2)} = \{u_3\}$. It is clear that $A_{(\alpha_2,\beta_2,\gamma_2)} \subseteq A_{(\alpha_1,\beta_1,\gamma_1)}$.
2.$A_{\overline{(\alpha_1,\beta_1,\gamma_1)}} = \{u_1, u_2, u_3\}$ and $A_{\overline{(\alpha_2,\beta_2,\gamma_2)}} = \{u_2\}$.Therefore $A_{\overline{(\alpha_2,\beta_2,\gamma_2)}} \subseteq A_{\overline{(\alpha_1,\beta_1,\gamma_1)}}$.

Definition 3.9. *Let* $A = \{< u, (T_A^1(u), T_A^2(u), ..., T_A^P(u)), (I_A^1(u), I_A^2(u), ..., I_A^P(u)), (F_A^1(u), F_A^2(u), ..., F_A^P(u)) >: u \in U\}, \{T_A^i(u), I_A^i(u), F_A^i(u) \in [0, 1]\}, (i \in \{1, 2, ..., p\})$ *be any neutrosophic multi-sets on U. Then* α*-cut,* β*-cut,* γ*-cut,* (α, β)*-cut,* (α, γ)*-cut and* (β, γ)*-cut set of neutrosophic multi-sets A is denoted by* A_α, A_β, A_γ, $A_{(\alpha,\beta)}$, $A_{(\alpha,\gamma)}$, $A_{(\beta,\gamma)}$. *respectively, is defined by;*

$$A_\alpha = \{u : T_A(u) \geqslant \alpha, u \in U\},$$

$$A_\beta = \{u : I_A(u) \leqslant \beta, u \in U\},$$

$$A_\gamma = \{u : F_A(u) \leqslant \gamma, u \in U\},$$

$$A_{(\alpha,\beta)} = \{u : T_A(u) \geqslant \alpha, I_A(u) \leqslant \beta, u \in U\},$$

$$A_{(\alpha,\gamma)} = \{u : T_A(u) \geqslant \alpha, F_A(u) \leqslant \gamma, u \in U\},$$

$$A_{(\beta,\gamma)} = \{u : I_A(u) \leqslant \beta, F_A(u) \leqslant \gamma, u \in U\}.$$

Example 3.10. Suppose that $A = \{u_1 : (0.4, 0.3, 0.5, 0.2), (0.5, 0.2, 0.4, 0.3), (0.3, 0.6, 0.5, 0.2), u_2 : (0.8, 0.4, 0.3, 0.6), (0.4, 0.3, 0.6, 0.1), (0.7, 0.3, 0.4, 0.2), u_3 : (0.7, 0.6, 0.5, 0.8), (0.1, 0.6, 0.4, 0.2), (0.5, 0.4, 0.6, 0.8)\}$. Let $(\alpha = 0.6, \beta = 0.4, \gamma = 0.3) \in [0, 1]$ Then,

$$A_\alpha = \{u_2, u_3\},$$
$$A_\beta = \{u_1, u_2, u_3\},$$
$$A_\gamma = \{u_1, u_2\},$$
$$A_{(\alpha,\beta)} = \{u_2, u_3\},$$
$$A_{(\alpha,\gamma)} = \{u_2\},$$
$$A_{(\beta,\gamma)} = \{u_1, u_2\}.$$

Theorem 3.11. *Let* $A = \{< u, (T^1_A(u), T^2_A(u), ..., T^P_A(u)), (I^1_A(u), I^2_A(u), ..., I^P_A(u)), (F^1_A(u), F^2_A(u), ..., F^P_A(u)) >: u \in U\}, \{T^i_A(u), I^i_A(u), F^i_A(u) \in [0,1]\}, (i \in \{1, 2, ..., p\})$ *be any neutrosophic multi-sets on* U *and* $A_{(\alpha,\beta,\gamma)}$ *be a set of* (α, β, γ)*-cut. Then,*

1. $A_{(\alpha,\beta)} = A_\alpha \cap A_\beta$.
2. $A_{(\beta,\gamma)} = A_\beta \cap A_\gamma$.
3. $A_{(\gamma,\gamma)} = A_\gamma \cap A_\gamma$.
4. $A_{(\alpha,\beta,\gamma)}=A_\alpha \cap A_{(\beta,\gamma)}$.
5. $A_{(\alpha,\beta,\gamma)}=A_{(\alpha,\gamma)} \cap A_\beta$.
6. $A_{(\alpha,\beta,\gamma)}=A_{(\alpha,\beta)} \cap A_\gamma$.
7. $A_{(\alpha,\beta,\gamma)}=A_\alpha \cap A_\beta \cap A_\gamma$.

Proof. It is trivial. □

Theorem 3.12. *Let* A, B, A_i *and* $B_i (i \in I)$ *be any neutrosophic multi-sets on* U. *Then,*

1. $(A \cup B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma).}$
2. $(A \cup B)_{\overline{(\alpha,\beta,\gamma)}} = A_{\overline{(\alpha,\beta,\gamma)}} \cup B_{\overline{(\alpha,\beta,\gamma).}}$
3. $\bigcup_i (A_i)_{(\alpha,\beta,\gamma)} \subseteq (\bigcup_i A_i)_{(\alpha,\beta,\gamma).}$
4. $\bigcup_i (A_i)_{\overline{(\alpha,\beta,\gamma)}} = (\bigcup_i A_i)_{\overline{(\alpha,\beta,\gamma).}}$

Proof.
1. Let A and B be any neutrosophic multisets on U. Then,

$$\begin{aligned} A\cup B_{(\alpha,\beta,\gamma)} &= \{u : T^i_{A\cup B}(u) \geqslant \alpha, I^i_{A\cup B}(u) \leqslant \beta, F^i_{A\cup B}(u) \leqslant \gamma, u \in U\} \\ &= \{u : T^i_{A\cup B}(u) \wedge \alpha = \alpha, I^i_{A\cup B}(u) \vee \beta = \beta, F^i_{A\cup B}(u) \vee \gamma = \gamma, u \in U\} \\ &= \{u : (T^i_A(u) \vee T^i_B(u)) \wedge \alpha = \alpha, (I^i_A(u) \vee I^i_B(u)) \vee \beta = \beta \\ &\quad , (F^i_A(u) \vee F^i_A(u)) \vee \gamma = \gamma, u \in U\} \\ &= \{u : T^i_A(u) \wedge \alpha = \alpha, I^i_A(u) \vee \beta = \beta, F^i_A(u) \vee \gamma = \gamma, u \in U\} \vee \\ &\quad \{u : T^i_B(u) \wedge \alpha = \alpha, I^i_B(u) \vee \beta = \beta, F^i_B(u) \vee \gamma = \gamma, u \in U\} \\ &= A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}. \end{aligned}$$

2. Let A and B be any neutrosophic multisets on U. Then,

$$\begin{aligned} (A\cup B)_{\overline{(\alpha,\beta,\gamma)}} &= \{u : T^i_{A\cup B}(u) > \alpha, I^i_{A\cup B}(u) < \beta, F^i_{A\cup B}(u) < \gamma, u \in U\} \\ &= \{u : T^i_{A\cup B}(u) \wedge \alpha = \alpha, I^i_{A\cup B}(u) \vee \beta = \beta, F^i_{A\cup B}(u) \vee \gamma = \gamma, u \in U\} \\ &= \{u : (T^i_A(u) \vee T^i_B(u)) \wedge \alpha = \alpha, (I^i_A(u) \vee I^i_B(u)) \vee \beta = \beta \\ &\quad , (F^i_A(u) \vee F^i_A(u)) \vee \gamma = \gamma, u \in U\} \\ &= \{u : T^i_A(u) \wedge \alpha = \alpha, I^i_A(u) \vee \beta = \beta, F^i_A(u) \vee \gamma = \gamma, u \in U\} \vee \\ &\quad \{u : T^i_B(u) \wedge \alpha = \alpha, I^i_B(u) \vee \beta = \beta, F^i_B(u) \vee \gamma = \gamma, u \in U\} \\ &= A_{\overline{(\alpha,\beta,\gamma)}} \cup B_{\overline{(\alpha,\beta,\gamma)}}. \end{aligned}$$

The proofs of (3) and (4) can be made similarly. □

Theorem 3.13. *Let A, B, A_i and $B_i (i \in I)$ be any neutrosophic multisets on U. Then,*

1. $(A \cap B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma).}$
2. $(A \cap B)_{\overline{(\alpha,\beta,\gamma)}} = A_{\overline{(\alpha,\beta,\gamma)}} \cap B_{\overline{(\alpha,\beta,\gamma).}}$
3. $\bigcap_i (A_i)_{(\alpha,\beta,\gamma)} \subseteq (\bigcap_i A_i)_{(\alpha,\beta,\gamma).}$
4. $\bigcap_i (A_i)_{\overline{(\alpha,\beta,\gamma)}} = (\bigcap_i A_i)_{\overline{(\alpha,\beta,\gamma).}}$

Proof.
1. We can see that

$$\begin{aligned} A\cap B_{(\alpha,\beta,\gamma)} &= \{u : T^i_{A\cap B}(u) \geqslant \alpha, I^i_{A\cap B}(u) \leqslant \beta, F^i_{A\cap B}(u) \leqslant \gamma, u\in U\} \\ &= \{u : T^i_{A\cap B}(u)\wedge\alpha=\alpha, I^i_{A\cap B}(u)\vee\beta=\beta, F^i_{A\cap B}(u)\vee\gamma=\gamma, u\in U\} \\ &= \{u : (T^i_A(u)\wedge T^i_B(u))\wedge\alpha=\alpha, (I^i_A(u)\wedge I^i_B(u))\vee\beta=\beta \\ &\quad ,(F^i_A(u)\wedge F^i_A(u))\vee\gamma=\gamma, u\in U\} \\ &= \{u : T^i_A(u)\wedge\alpha=\alpha, I^i_A(u)\vee\beta=\beta, F^i_A(u)\vee\gamma=\gamma, u\in U\}\wedge \\ &\quad \{u : T^i_B(u)\wedge\alpha=\alpha, I^i_B(u)\vee\beta=\beta, F^i_B(u)\vee\gamma=\gamma, u\in U\} \\ &= A_{(\alpha,\beta,\gamma)}\cap B_{(\alpha,\beta,\gamma)}. \end{aligned}$$

2. We have

$$\begin{aligned} (A\cap B)_{\overline{(\alpha,\beta,\gamma)}} &= \{u : T^i_{A\cap B}(u) > \alpha, I^i_{A\cap B}(u) < \beta, F^i_{A\cap B}(u) < \gamma, u\in U\} \\ &= \{u : T^i_{A\cap B}(u)\wedge\alpha=\alpha, I^i_{A\cap B}(u)\vee\beta=\beta, F^i_{A\cap B}(u)\vee\gamma=\gamma, u\in U\} \\ &= \{u : (T^i_A(u)\wedge T^i_B(u))\wedge\alpha=\alpha, (I^i_A(u)\wedge I^i_B(u))\vee\beta=\beta \\ &\quad ,(F^i_A(u)\wedge F^i_A(u))\vee\gamma=\gamma, u\in U\} \\ &= \{u : T^i_A(u)\wedge\alpha=\alpha, I^i_A(u)\vee\beta=\beta, F^i_A(u)\vee\gamma=\gamma, u\in U\}\wedge \\ &\quad \{u : T^i_B(u)\wedge\alpha=\alpha, I^i_B(u)\vee\beta=\beta, F^i_B(u)\vee\gamma=\gamma, u\in U\} \\ &= A_{\overline{(\alpha,\beta,\gamma)}}\cap B_{\overline{(\alpha,\beta,\gamma)}}. \end{aligned}$$

The proofs of (3) and (4) can be made similarly. □

Theorem 3.14. *Let A be any neutrosophic multi-sets on U and for $i\in I$, $(\alpha_1,\beta_1,\gamma_1) = max_i\{(\alpha_i,\beta_i,\gamma_i)\}$ and $(\alpha_2,\beta_2,\gamma_2) = min_i\{(\alpha_i,\beta_i,\gamma_i)\}$,*

1. $\bigcap_i (A_i)_{(\alpha_i,\beta_i,\gamma_i)} = A_{(\alpha_1,\beta_1,\gamma_1).}$
2. $\bigcup_i (A_i)_{\overline{(\alpha_i,\beta_i,\gamma_i)}} = A_{\overline{(\alpha_2,\beta_2,\gamma_2).}}$

Proof.

1. Note that

$$\begin{aligned} \textstyle\bigcap_i (A_i)_{(\alpha_i,\beta_i,\gamma_i)} &= \{u : \textstyle\bigvee_i T^i_A(u)\geqslant\alpha_i, \bigwedge_i I^i_A(u)\leqslant\beta_i, \bigwedge_i F^i_A(u)\leqslant\gamma_i, u\in U\} \\ &= \{u : \textstyle\bigvee_i T^i_A(u)\wedge\alpha_i=\alpha_1, \bigwedge_i I^i_A(u)\vee\beta_i=\beta_1, \bigwedge_i F^i_A(u)\vee\gamma_i=\gamma_1, u\in U\} \\ &= \{u : (T^i_A(u)\wedge T^i_B(u))\wedge\alpha_1=\alpha_1, (I^i_A(u)\wedge I^i_B(u))\vee\beta_1=\beta_1, \\ &\quad (F^i_A(u)\wedge F^i_A(u))\vee\gamma_1=\gamma_1, u\in U\} \\ &= A_{(\alpha_1,\beta_1,\gamma_1).} \end{aligned}$$

2.

$$\begin{aligned}\bigcup_i (A_i)_{\overline{(\alpha_i,\beta_i,\gamma_i)}} &= \{u : \bigwedge_i T_A^i(u) > \alpha_i, \bigvee_i I_A^i(u) < \beta_i, \bigvee_i F_A^i(u) < \gamma_i, u \in U\} \\ &= \{u : \bigwedge_i T_A^i(u) \wedge \alpha_i = \alpha_2, \bigvee_i I_A^i(u) \vee \beta_i = \beta_2, \bigvee_i F_A^i(u) \vee \gamma_i = \gamma_2, u \in U\} \\ &= \{u : (T_A^i(u) \wedge T_B^i(u)) \wedge \alpha_2 = \alpha_2, (I_A^i(u) \wedge I_B^i(u)) \vee \beta_2 = \beta_2, \\ &(F_A^i(u) \wedge F_A^i(u)) \vee \gamma_2 = \gamma_2, u \in U\} \\ &= A_{\overline{(\alpha_2,\beta_2,\gamma_2)}}. \quad \square\end{aligned}$$

Definition 3.15. *Let A be any two neutrosophic multi-sets on U and for $(\alpha, \beta, \gamma) \in [0, 1]^3$ $(\alpha, \beta, \gamma)A$ is defined as; $\forall u \in U$,*

$$(\alpha, \beta, \gamma)A(u) = \{u : T_A^i(u) \wedge \alpha, I_A^i(u) \vee \beta, F_A^i(u) \vee \gamma, u \in U\}.$$

Even though A is a crisp set, $(\alpha, \beta, \gamma)A(u)$ is still a neutrosophic multi-sets defined by

$$(\alpha, \beta, \gamma)A(u) = \begin{cases} (\alpha, \beta, \gamma), & u \in A \\ (0, 0, 0), & u \notin A. \end{cases}$$

Theorem 3.16. *Let A, B be any neutrosophic multi-sets on U and for $(\alpha_i, \beta_i, \gamma_i) \in [0, 1]^3$ $(i = 1, 2)$. We have*

1. *If $(\alpha_1, \beta_1, \gamma_1) \leqslant (\alpha_2, \beta_2, \gamma_2)$, then $(\alpha_1, \beta_1, \gamma_1)A \subseteq (\alpha_2, \beta_2, \gamma_2)A$*
2. *If $A \subseteq B$, then $(\alpha_1, \beta_1, \gamma_1)A \subseteq (\alpha_1, \beta_1, \gamma_1)B$*

Proof. Let A and B be any neutrosophic multisets on U and for $(\alpha_i, \beta_i, \gamma_i)$ $\in [0, 1]^3 (i = 1, 2)$. We have
1. $(\alpha_1, \beta_1, \gamma_1) \leqslant (\alpha_2, \beta_2, \gamma_2) \Rightarrow$

$$\begin{aligned}(\alpha_1, \beta_1, \gamma_1)A &= \{u : T_A^i(u) \wedge \alpha_1, I_A^i(u) \vee \beta_1, F_A^i(u) \vee \gamma_1, u \in U\} \\ &\leqslant \{u : T_A^i(u) \wedge \alpha_2, I_A^i(u) \vee \beta_2, F_A^i(u) \vee \gamma_2, u \in U\} \\ &= (\alpha_2, \beta_2, \gamma_2)A.\end{aligned}$$

Therefore $(\alpha_1, \beta_1, \gamma_1)A \subseteq (\alpha_2, \beta_2, \gamma_2)A$.
2. If $A \subseteq B$ *then* ($A \subseteq B$ if $T_A(u) \leqslant T_B(u)$, $I_A(u) \geqslant I_B(u)$,$F_A(u) \geqslant$

$F_B(u)$, $\forall u \in U$)

$$\begin{aligned}(\alpha_1,\beta_1,\gamma_1)A &= \{u : T^i_A(u)\wedge\alpha_1, I^i_A(u)\vee\beta_1, F^i_A(u)\vee\gamma_1, u\in U\}\\ &\leqslant \{u : T^i_B(u)\wedge\alpha_1, I^i_B(u)\vee\beta_1, F^i_B(u)\vee\gamma_1, u\in U\}\\ &= (\alpha_1,\beta_1,\gamma_1)B.\end{aligned}$$

Therefore $(\alpha_1,\beta_1,\gamma_1)A \subseteq (\alpha_1,\beta_1,\gamma_1)B$. □

Theorem 3.17. *Let A be any a neutrosophic multi-sets on U. Then,*

$$A = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma) A_{(\alpha,\beta,\gamma)}.$$

Proof. Let A be any neutrosophic multisets on U and for $(\alpha,\beta,\gamma)\in[0,1]^3$. We have

$$\begin{aligned}\bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3}(\alpha,\beta,\gamma)A_{(\alpha,\beta,\gamma)} &= \bigvee_{(\alpha,\beta,\gamma)\in[0,1]^3}\{(\alpha,\beta,\gamma)\wedge A_{(\alpha,\beta,\gamma)}\}\\ &= \bigvee_{(\alpha,\beta,\gamma)\in[0,1]^3}\{(\alpha,\beta,\gamma) : T^i_A(u)\geqslant\alpha,\\ &\quad I^i_A(u)\leqslant\beta, F^i_A(u)\leqslant\gamma, u\in U\}\\ &= (T^i_A(u), I^i_A(u), F^i_A(u))\\ &= A.\end{aligned}$$

Theorem 3.18. *Let A be any a neutrosophic multi-sets on U. Then,*

$$A = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma) A_{\overline{(\alpha,\beta,\gamma)}}.$$

Proof. Let A be any neutrosophic multisets on U and for $(\alpha,\beta,\gamma)\in[0,1]^3$. We have

$$\begin{aligned}\bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3}(\alpha,\beta,\gamma)A_{\overline{(\alpha,\beta,\gamma)}} &= \bigvee_{(\alpha,\beta,\gamma)\in[0,1]^3}\{(\alpha,\beta,\gamma)\wedge A_{\overline{(\alpha,\beta,\gamma)}}\}\\ &= \bigvee_{(\alpha,\beta,\gamma)\in[0,1]^3}\{(\alpha,\beta,\gamma) : T^i_A(u)>\alpha,\\ &\quad I^i_A(u)<\beta, F^i_A(u)<\gamma, u\in U\}\\ &= (T^i_A(u), I^i_A(u), F^i_A(u))\\ &= A.\end{aligned}$$

Theorem 3.19. *Let A, B be any two neutrosophic multi-sets on U. Then,*

$$\begin{aligned}A = B &\Leftrightarrow \forall(\alpha,\beta,\gamma)\in[0,1]^3, A_{(\alpha,\beta,\gamma)} = B_{(\alpha,\beta,\gamma)}\\ &\Leftrightarrow \forall(\alpha,\beta,\gamma)\in[0,1]^3, A_{\overline{(\alpha,\beta,\gamma)}} = B_{\overline{(\alpha,\beta,\gamma)}}\end{aligned}$$

Proof. Let A and B be any neutrosophic multisets on U and for $(\alpha,\beta,\gamma)\in[0,1]^3$. We hav
$A = B(A = B$ if $T_A(u) = T_B(u)$, $I_A(u) = I_B(u)$,$F_A(u) = F_B(u)$, $\forall u \in U)\Leftrightarrow$

$$\begin{aligned} A_{(\alpha,\beta,\gamma)} &= \{u : T_A^i(u) \geqslant \alpha, I_A^i(u) \leqslant \beta, F_A^i(u) \leqslant \gamma, u \in U\} \\ &= \{u : T_A^i(u) \wedge \alpha, I_A^i(u) \vee \beta, F_A^i(u) \vee \gamma, u \in U\} \\ &= \{u : T_B^i(u) \wedge \alpha, I_B^i(u) \vee \beta, F_B^i(u) \vee \gamma, u \in U\} \\ &= B_{(\alpha,\beta,\gamma)}. \end{aligned}$$

Therefore $A_{(\alpha,\beta,\gamma)} = B_{(\alpha,\beta,\gamma)}$.
$A = B(A = B$ if $T_A(u) = T_B(u)$, $I_A(u) = I_B(u)$,$F_A(u) = F_B(u)$, $\forall u \in U)\Leftrightarrow$

$$\begin{aligned} A_{\overline{(\alpha,\beta,\gamma)}} &= \{u : T_A^i(u) > \alpha, I_A^i(u) < \beta, F_A^i(u) < \gamma, u \in U\} \\ &= \{u : T_A^i(u) \wedge \alpha, I_A^i(u) \vee \beta, F_A^i(u) \vee \gamma, u \in U\} \\ &= \{u : T_B^i(u) \wedge \alpha, I_B^i(u) \vee \beta, F_B^i(u) \vee \gamma, u \in U\} \\ &= B_{\overline{(\alpha,\beta,\gamma)}}. \end{aligned}$$

Therefore $A_{\overline{(\alpha,\beta,\gamma)}} = B_{\overline{(\alpha,\beta,\gamma)}}$.

Theorem 3.20. *Let A be any a neutrosophic multi-sets on U and for $\delta = (\alpha,\beta,\gamma)\in[0,1]^3$. If $H : [0,1]^3 \to P(X)$ satisfies that $\forall\delta \in [0,1]^3, A_{\overline{\delta}} \subseteq H(\delta) \subseteq A_\delta$, then*

$$A = \bigcup_{\delta\in[0,1]^3} \delta H(\delta)$$

and the following properties are fulfilled:

1. *$H(\delta_1) \supseteq H(\delta_2)$ whenever $\delta_1 < \delta_2$.*
2. *$\forall\delta \in [0,1]^3, A_\delta = \bigcap_{\theta<\delta} H(\theta)$.*
3. *$\forall\delta \in [0,1]^3, A_{\overline{\delta}} = \bigcap_{\theta>\delta} H(\theta)$.*

Proof.
1. It is obvious.
2. When $\theta < \delta$, we have $H(\theta) \supset A_{\overline{\theta}} \supset A_\delta$. Thus $\bigcap_{\theta<\delta} H(\theta) \supset A_\delta$. On the other hand $\bigcap_{\theta<\delta} H(\theta) \subset \bigcap_{\theta<\delta} A_\theta = A_\delta$. Hence $A_\delta = \bigcap_{\theta<\delta} H(\theta)$.

3. When $\theta > \delta$, we have $H(\theta) \subset A_\theta \subset A_{\overline{\delta}}$. Thus $A_{\overline{\delta}} \supset \bigcap_{\theta>\delta} H(\theta)$. On the other hand,$H(\theta) \subset A_{\overline{\theta}}$ and $\bigcap_{\theta>\delta} H_\theta \supset \bigcap_{\theta>\delta} A_{\overline{\theta}} = A_{\overline{\delta}}$. Therefore, $A_{\overline{\delta}} = \bigcap_{\theta>\delta} H(\theta)$.
Let extending the function $f : X \to Y$. we recall the (direct)image of a the neutrosophic multi-subsets A of X under f is defined as

$$f(A) = \{T^i_{f(A)}, I^i_{f(A)}, F^i_{f(A)}\} \subseteq Y,$$

and the inverse image of a neutrosophic multi-subsets B of Y is defined as

$$f^{-1}(A) = \{T^i_{f(A)}, I^i_{f(A)}, F^i_{f(A)}\} \subseteq X.$$

In other words, the mapping $f : X \to Y$ induces two new mappings $f : P(X) \to P(Y)$ and $f^{-1} : P(Y) \to P(X)$. These induced mappings possess the following properties.

Theorem 3.21. $f : X \to Y, A, B, A_i \in P(X)(i \in J)$, *then the induced mappings f satisfies that*

1. $A \subseteq B$ *implies* $f(A) \subseteq f(B)$.
2. $f(\bigcup_i(A_i)) = (\bigcup_i f(A_i))$.
3. $f(\bigcap_i(A_i)) \subseteq (\bigcap_i f(A_i))$.
4. $\forall y \in Y (f(A))(y) = \bigvee_{T^i_{f(A)}(x)=y} A(x), \bigwedge_{I^i_{f(A)}(x)=y} A(x), \bigwedge_{F^i_{f(A)}(x)=y} A(x)$.
5. $\forall(\alpha,\beta,\gamma) \in [0,1]^3, f(A_{(\alpha,\beta,\gamma)}) \subseteq (f(A))_{(\alpha,\beta,\gamma)}$.
6. $f(A) = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3}(\alpha,\beta,\gamma)(f(A))_{(\alpha,\beta,\gamma)} = \bigcup_{(\alpha,\beta,\gamma)\in.[0,1]^3}(\alpha,\beta,\gamma)f(A_{(\alpha,\beta,\gamma)})$.

Proof.
1. It is Trivial.
2. Let $y \in Y$. Then

$$\begin{aligned} y \in f(\textstyle\bigcup_i(A_i)) &\Leftrightarrow \exists x \in \textstyle\bigcup_i(A_i), \{T^i_{f(A)}(x) = y, I^i_{f(A)}(x) = y, F^i_{f(A)}(x) = y\} \\ &\Leftrightarrow \exists i \in J, x \in (A_i) and \{T^i_{f(A)}(x) = y, I^i_{f(A)}(x) = y, F^i_{f(A)}(x) = y\} \\ &\Leftrightarrow \exists i \in J, y \in f(A_i) \\ &\Leftrightarrow y \in \textstyle\bigcup_i f(A_i). \end{aligned}$$

3. It is similar to the proof of (2)
4. Let $y \in Y$. Then

$$
\begin{aligned}
(f(A))(y) = \{1,0,0\} \\
&\Leftrightarrow y \in f(A) \\
&\Leftrightarrow \exists x \in A, \{T^i_{f(A)}(x) = y, I^i_{f(A)}(x) = y, F^i_{f(A)}(x) = y\} \\
&\Leftrightarrow \exists x \in A, A(x) = \{1,0,0\} and \{T^i_{f(A)}(x) = y, I^i_{f(A)}(x) = y, F^i_{f(A)}(x) = y\} \\
&\Leftrightarrow \bigvee_{T^i_{f(A)}(x)=y} A(x) = 1, \bigwedge_{I^i_{f(A)}(x)=y} A(x) = 0, \bigwedge_{F^i_{f(A)}(x)=y} A(x) = 0.
\end{aligned}
$$

5. Let $y \in Y$. Then

$$
\begin{aligned}
y \in (f(A))_{(\alpha,\beta,\gamma)} &\Leftrightarrow T^i_{f(A)}(y) > \alpha, I^i_{f(A)}(y) < \beta, F^i_{f(A)}(y) < \gamma \\
&\Leftrightarrow \bigvee_{T^i_{f(A)}(x)=y} A(x) > \alpha, \bigwedge_{I^i_{f(A)}(x)=y} A(x) < \beta, \bigwedge_{F^i_{f(A)}(x)=y} A(x) < \gamma \\
&\Leftrightarrow \exists x \in X, T^i_{f(A)}(x) = y, I^i_{f(A)}(x) = y, F^i_{f(A)}(x) = y, A(x) > (\alpha,\beta,\gamma) \\
&\Leftrightarrow \exists x \in X A_{(\alpha,\beta,\gamma)}, T^i_{f(A)}(x) = y, I^i_{f(A)}(x) = y, F^i_{f(A)}(x) = y \\
&\Leftrightarrow y \in f(A_{(\alpha,\beta,\gamma)}).
\end{aligned}
$$

6. By for every $A \in P(X), A = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma) A_{(\alpha,\beta,\gamma)}$ and (5)

$$
f(A) = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma)(f(A))_{(\alpha,\beta,\gamma)} = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma) f(A_{(\alpha,\beta,\gamma)}).
$$

For every $y \in Y$, we have

$$
\begin{aligned}
\bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma)(f(A))_{(\alpha,\beta,\gamma)} &= \bigvee_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma) \wedge (f(A_{(\alpha,\beta,\gamma)}))(y) \\
&= \bigvee_{\alpha\in[0,1]} (\alpha \wedge \bigvee_{T^i_{f(A)}(x)=y} A_\alpha(x), \\
&\bigwedge_{\beta\in[0,1]} (\beta \vee \bigwedge_{I^i_{f(A)}(x)=y} A_\beta(x)), \\
&\bigwedge_{\gamma\in[0,1]} (\gamma \vee \bigwedge_{F^i_{f(A)}(x)=y} A_\gamma(x)) \\
&= \bigvee_{T^i_{f(A)}(x)=y} \bigvee_{\alpha\in[0,1]} \alpha \wedge A_\alpha(x), \\
&\bigwedge_{I^i_{f(A)}(x)=y} \bigwedge_{\beta\in[0,1]} \beta \vee A_\beta(x), \\
&\bigwedge_{F^i_{f(A)}(x)=y} \bigwedge_{\gamma\in[0,1]} \gamma \vee A_\gamma(x) \\
&= \bigvee_{T^i_{f(A)}(x)=y} A(x), \bigwedge_{I^i_{f(A)}(x)=y} A(x), \\
&\bigwedge_{F^i_{f(A)}(x)=y} A(x) \\
&= (f(A))(y).
\end{aligned}
$$

Thus $f(A) = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3} (\alpha,\beta,\gamma) f(A_{(\alpha,\beta,\gamma)})$.

Theorem 3.22. *$f: X \to Y A, B, B_i \in P(Y)(i \in J)$, then the induced mappings f^{-1} satisfies that*

1. *$A \subseteq B$ implies $f^{-1}(A) \subseteq f^{-1}(B)$.*
2. *$f^{-1}(B^c) = (f^{-1}(B))^c$.*
3. *$f^{-1}(\bigcup_i(B_i)) = (\bigcup_i f^{-1}(B_i))$.*
4. *$f^{-1}(\bigcap_i(B_i)) = (\bigcap_i f^{-1}(B_i))$.*
5. *$\forall(\alpha,\beta,\gamma) \in [0,1]^3, f^{-1}(B_{(\alpha,\beta,\gamma)}) = (f^{-1}(B))_{(\alpha,\beta,\gamma)}$.*
6. *$f^{-1}(B) = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3}(\alpha,\beta,\gamma)(f^{-1}(B))_{(\alpha,\beta,\gamma)} = \bigcup_{(\alpha,\beta,\gamma)\in[0,1]^3}(\alpha,\beta,\gamma)f^{-1}(B_{(\alpha,\beta,\gamma)})$.*

Proof. We proof (4) and (5) as examples.
4.

$$\begin{aligned}\forall x \in X, f^{-1}(\textstyle\bigcap_i(B_i))(x) &= \textstyle\bigcap_i(B_i)(f(x)) \\ &= \textstyle\bigwedge_i(B_i)(T^i_{f(A)}(x)), \bigvee_i(B_i)(I^i_{f(A)}(x)), \bigvee_i(B_i)(F^i_{f(A)}(x)) \\ &= \textstyle\bigwedge f^{-1}(B_i)(x) \\ &= (\textstyle\bigcap_i f^{-1}(B_i))(x).\end{aligned}$$

5.

$$\begin{aligned}x \in f^{-1}(B_{(\alpha,\beta,\gamma)}) &\Leftrightarrow \{T^i_{f(A)}(x), I^i_{f(A)}(x), F^i_{f(A)}(x)\} \in B_{(\alpha,\beta,\gamma)} \\ &\Leftrightarrow B(T^i_{f(A)}(x)) \geqslant \alpha, B(I^i_{f(A)}(x)) \leqslant \beta, B(F^i_{f(A)}(x)) \leqslant \gamma \\ &\Leftrightarrow (f^{-1}(B))(x) \geqslant (\alpha,\beta,\gamma) \\ &\Leftrightarrow x \in (f^{-1}(B))_{(\alpha,\beta,\gamma)}.\end{aligned}$$

The proofs of (1),(2),(3) and (6) can be made similarly. □

4. Algebraic Operations Over Neutrosophic Multi-Sets Based on Extension Principle

Let A and B be two neutrosophic multi-sets on the universal sets X and Y respectively. According to the extension principle of the neutrosophic multi-sets. We can define algebraic operations over the neutrosophic multi-sets A and B, i.e.,the addition, subtraction, multiplication, and division operations are defined as follows:

$$A * B = \{\langle z, \vee_{z=x*y}\{T^i_{f(A)}(x) \wedge T^i_{f(B)}(x)\}, \wedge_{z=x*y}\{I^i_{f(A)}(x) \vee I^i_{f(B)}(x)\},$$
$$\wedge_{z=x*y}\{F^i_{f(A)}(x) \vee F^i_{f(B)}(x)\}\rangle | (x,y) \in X \times Y\},$$

where the symbol "$*$" represent one of the algebraic operations "$+$", "$-$", "$\times$", and "$\div$". The division operation "$\div$" is required to satisfy the condition: $0 \in supp(B) = \{y | T^i_{f(B)}(y) \geqslant 0, I^i_{f(B)}(y) \leqslant 1, F^i_{f(B)}(y) \leqslant 1, y \in Y$.
More specifically, $A+B, A-B, A\times B, A\div B$ and $\lambda \times B$ are defined as follows:

$$\begin{aligned} A+B = \ & \{\langle z, \vee_{z=x+y}\{T^i_{f(A)}(x) \wedge T^i_{f(B)}(y)\}, \wedge_{z=x+y}\{I^i_{f(A)}(x) \vee I^i_{f(B)}(y)\}, \\ & \wedge_{z=x+y}\{F^i_{f(A)}(x) \vee F^i_{f(B)}(y)\}\rangle | (x,y) \in X \times Y\} \end{aligned}$$

$$\begin{aligned} A-B = \ & \{\langle z, \vee_{z=x-y}\{T^i_{f(A)}(x) \wedge T^i_{f(B)}(y)\}, \wedge_{z=x-y}\{I^i_{f(A)}(x) \vee I^i_{f(B)}(y)\}, \\ & \wedge_{z=x-y}\{F^i_{f(A)}(x) \vee F^i_{f(B)}(y)\}\rangle | (x,y) \in X \times Y\} \end{aligned}$$

$$\begin{aligned} A\times B = \ & \{\langle z, \vee_{z=x\times y}\{T^i_{f(A)}(x) \wedge T^i_{f(B)}(y)\}, \wedge_{z=x\times y}\{I^i_{f(A)}(x) \vee I^i_{f(B)}(y)\}, \\ & \wedge_{z=x\times y}\{F^i_{f(A)}(x) \vee F^i_{f(B)}(y)\}\rangle | (x,y) \in X \times Y\} \end{aligned}$$

$$\begin{aligned} A\div B = \ & \{\langle z, \vee_{z=x\div y}\{T^i_{f(A)}(x) \wedge T^i_{f(B)}(y)\}, \wedge_{z=x\div y}\{I^i_{f(A)}(x) \vee I^i_{f(B)}(y)\}, \\ & \wedge_{z=x\div y}\{F^i_{f(A)}(x) \vee F^i_{f(B)}(y)\}\rangle | (x,y) \in X \times Y\} \end{aligned}$$

$$\begin{aligned} \lambda B = \ & \{\langle z, \vee_{z=\lambda\times y}\{T^i_{f(B)}(y)\}, \wedge_{z=\lambda\times y}\{I^i_{f(B)}(y)\}, \\ & \wedge_{z=\lambda\times y}\{F^i_{f(B)}(y)\}\rangle | (x,y) \in Y\}, \end{aligned}$$

respectively, where $\lambda \neq 0$ is any real number.

Example 4.1. Let A and B be two neutrosophic multi-set as follows;

$$\begin{aligned} A = \ & \{\langle 1, (0.5, 0.6, 0.8, 0.7), (0.2, 0.1, 0.3, 0.5), (0.3, 0.4, 0.1, 0.5)\rangle, \langle 2, (0.6, 0.4, 0.7, 0.9), \\ & (0.4, 0.3, 0.5, 0.2), (0.2, 0.1, 0.4, 0.3)\rangle, \langle 3, (1, 0.8, 0.5, 0.7), (0.1, 0.2, 0.4, 0.3), \\ & (0.4, 0.3, 0.2, 0.1)\rangle\}. \end{aligned}$$

and

$$\begin{aligned} B = \ & \{\langle 4, (0.8, 0.5, 0.6, 0.7), (0.3, 0.1, 0.2, 0.4), (0.6, 0.4, 0.5, 0.7)\rangle, \langle 2, (0.7, 0.6, 0.4, 0.8), \\ & (0.5, 0.3, 0.4, 0.2), (0.3, 0.7, 0.4, 0.6)\rangle\}. \end{aligned}$$

respectively. Then,

$$\begin{aligned}A+B= &\{\langle 3,(0.5,0.6,0.4,0.8),(0.5,0.3,0.4,0.5),(0.3,0.7,0.4,0.6)\rangle,\langle 4,(0.6,0.4,0.4,0.8),\\ &(0.5,0.3,0.5,0.2),(0.3,0.7,0.4,0.6)\rangle,\langle 5,(0.7,0.6,0.4,0.7),(0.5,0.3,0.4,0.3),\\ &(0.4,0.7,0.4,0.6)\rangle,\langle 6,(0.6,0.4,0.6,0.7),(0.4,0.3,0.5,0.4),(0.6,0.4,0.5,0.7)\rangle,\\ &\langle 7,(0.8,0.5,0.5,0.7),(0.3,0.2,0.4,0.4),(0.6,0.4,0.5,0.7)\rangle\}.\end{aligned}$$

$$\begin{aligned}A-B= &\{\langle -3,(0.5,0.5,0.6,0.7),(0.3,0.1,0.3,0.4),(0.6,0.4,0.5,0.7)\rangle,\langle -2,(0.6,0.4,0.6,0.7),\\ &(0.4,0.3,0.5,0.4),(0.6,0.4,0.5,0.7)\rangle,\langle -1,(0.8,0.5,0.5,0.7),(0.3,0.2,0.4,0.4),\\ &(0.6,0.4,0.5,0.7)\rangle,\langle 0,(0.6,0.4,0.4,0.8),(0.5,0.3,0.5,0.2),(0.3,0.7,0.4,0.6)\rangle,\\ &\langle 1,(0.7,0.6,0.4,0.7),(0.5,0.3,0.4,0.3),(0.4,0.7,0.4,0.6)\rangle\}.\end{aligned}$$

$$\begin{aligned}A\times B= &\{\langle 2,(0.5,0.6,0.4,0.8),(0.5,0.3,0.4,0.5),(0.3,0.7,0.4,0.6)\rangle,\langle 4,(0.6,0.4,0.4,0.8),\\ &(0.5,0.3,0.5,0.2),(0.3,0.7,0.4,0.6)\rangle,\langle 6,(0.7,0.6,0.4,0.7),(0.5,0.3,0.4,0.3),\\ &(0.4,0.7,0.4,0.6)\rangle,\langle 8,(0.6,0.4,0.6,0.7),(0.4,0.3,0.5,0.4),(0.6,0.4,0.5,0.7)\rangle,\\ &\langle 12,(0.8,0.5,0.5,0.7),(0.3,0.2,0.4,0.4),(0.6,0.4,0.5,0.7)\rangle\}.\end{aligned}$$

$$\begin{aligned}A\div B= &\{\langle 1/4,(0.5,0.5,0.6,0.7),(0.3,0.1,0.3,0.4),(0.6,0.4,0.5,0.7)\rangle,\langle 1/2,(0.5,0.6,0.4,0.8),\\ &(0.5,0.3,0.4,0.5),(0.3,0.7,0.4,0.6)\rangle,\langle 3/4,(0.8,0.5,0.5,0.7),(0.3,0.2,0.4,0.4),\\ &(0.6,0.4,0.5,0.7)\rangle,\langle 1,(0.6,0.4,0.4,0.8),(0.5,0.3,0.5,0.2),(0.3,0.7,0.4,0.6)\rangle,\\ &\langle 3/2,(0.7,0.6,0.4,0.7),(0.5,0.3,0.4,0.3),(0.4,0.7,0.4,0.6)\rangle\}.\end{aligned}$$

$$\begin{aligned}\lambda B= &\{\langle 4\lambda,(0.8,0.5,0.6,0.7),(0.3,0.1,0.2,0.4),(0.6,0.4,0.5,0.7)\rangle,\langle 2\lambda,(0.7,0.6,0.4,0.8),\\ &(0.5,0.3,0.4,0.2),(0.3,0.7,0.4,0.6)\rangle\}.\end{aligned}$$

Note that the algebraic operations defined in this section are remarkably different from those introduced by Ye [42]. The reason is thatYe [42] defined the operators over neutrosophic multiset on the basis of algebraic sum and product which is t-norm and t-conorm.

5. Conclusion

Using the concept of cut sets of neutrosophic multi-sets, we proposed the representation theorem for neutrosophic multi-sets. Based on the extension principle, we defined algebraic operations such as the addition, subtraction, multiplication and division, which were extensions of the operations over the intuitionistic fuzzy sets in [17]. The algebraic operations proposed in this paper were remarkably different from those defined by [11, 14, 42], which were not always rational from a logical point of view and practical aspects. Different applications of the representation theorem and extension principles of neutrosophic multi-sets will be examined in near future.

References

[1] S. Aggarwal, R. Biswas, and A. Q. Ansari, Neutrosophic Modeling and Control, Computer and Communication Technology, (2010), 718-723.

[2] S. A. Alblow, A. A. Salama, and M. Eisa, New Concepts of Neutrosophc Sets, *International Journal of Mathematics and Computer Applications Research,* 4 (1) (2014), 59-66.

[3] M. Arora and R. Biswas, Deployment of Neutrosophic Technology to Retrieve Answers for Queries Posed in Natural Language, *3. Int. Conf. on Comp. Sci. and Inform. Tech.*, (2010), 435-439.

[4] M. Arora, R. Biswas, and U. S. Pandy, Neutrosophic Relational Database Decomposition, *International Journal of Advanced Computer Science and Applications,* 2 (8) (2011), 121-125.

[5] A. Q. Ansari, R. Biswas, and S. Aggarwal, Proposal for Applicability of Neutrosophic Set Theory in Medical AI, *International Journal of Computer Applications,* 27 (5) (2011), 5-11.

[6] A. Q. Ansari, R. Biswas, and S. Aggarwal, Neutrosophic classifier:An extension of fuzzy classifer, Applied Soft Computing, 13 (2013), 563-573.

[7] K. T. Atanassov, Intuitionistic Fuzzy Set. *Fuzzy Sets and Systems,* 20 (1) (1986), 87-86.

[8] S. Broumi, I. Deli, and F. Smarandache, Relations on Interval Valued Neutrosophic Soft Sets, *Journal of New Results in Science,* 5 (2014), 1-20.

[9] S. Broumi and I. Deli, Correlation measure for neutrosophic refined sets and its application n medical Diagnosis, *Palestine journal of mathematics,* 5 (1) (2016), 135-143.

[10] S. Broumi, I. Deli, and F. Smarandache, (2015), Relations on neutrosophic multi sets with properties. http://arxiv.org/abs/1506.04025.

[11] R. Chatterjee, P. Majumdar, and S. K. Samanta, Single valued neutrosophic multisets, Annals of Fuzzy Mathematics and Informatics, x/x (2015) xx–xx.

[12] S. Das, M. B. Kar, and S. Kar, Group multi-criteria decision making using intuitionistic multi-fuzzy sets, *Journal of Uncertainty Analysis and Applications*, 10 (1) (2013), 1-16.

[13] I. Deli, Interval-valued neutrosophic soft sets and its decision making International Journal of Machine Learning and Cybernetics, DOI: 10.1007/s13042-015-0461-3.

[14] I. Deli, S. Broumi, and F. Smarandache, On neutrosophic multisets and its application in medical diagnosis, *Journal of New Theory,* 6 (2015), 88-98.

[15] I. Deli and S. Broumi, Neutrosophic Soft Matrices and NSM-decision Making, *Journal of Intelligent and Fuzzy Systems*, Doi: 10.3233/IFS-141505.

[16] P. A. Ejegwa and J. A. Awolola, Intuitionistic Fuzzy Multiset (IFMS) In Binomial Distributions, *International Journal Of Scientific and Technology Research*, 3 (4) (2014), 335-337.

[17] D. F. Li, Decision and Game Theory in Management With Intuitionistic Fuzzy Sets Studies in Fuzziness and Soft Computing Volume 308, springer, 2014.

[18] R. Muthuraj and S. Balamurugan, Multi-Fuzzy Group and its Level Subgroups, *Gen. Math. Notes*, 17 (1) (2013), 74-81.

[19] Z. Pawlak, Rough sets, *Int. J. Comput. Inform. Sci.,* 11 (1982), 341-356.

[20] P. Rajarajeswari and N. Uma, On Distance and Similarity Measures of Intuitionistic Fuzzy Multi Set, *IOSR Journal of Mathematics*, 5 (4) (2013), 19-23.

[21] P. Rajarajeswari and N. Uma, A Study of Normalized Geometric and Normalized Hamming Distance Measures in Intuitionistic Fuzzy Multi Sets, International Journal of Science and Research, *Engineering and Technology*, 2 (11) (2013), 76-80.

[22] P. Rajarajeswari and N. Uma, Normalized Hamming Similarity Measure for Intuitionistic Fuzzy Multi Sets and Its Application in Medical diagnosis, *International Journal of Mathematics Trends and Technology*, 5 (3) (2014), 219-225.

[23] P. Rajarajeswari and N. Uma, Intuitionistic Fuzzy Multi Relations, *International Journal of Mathematical Archives,* 4 (10) (2013), 244-249.

[24] P. Rajarajeswari and N. Uma, Zhang and Fu's Similarity Measure on Intuitionistic Fuzzy Multi Sets, International Journal of Innovative Research in Science, *Engineering and Technology,* 3 (5) (2014), 12309-12317.

[25] P. Rajarajeswari and N. Uma, Correlation Measure For Intuitionistic Fuzzy Multi Sets, *International Journal of Research in Engineering and Technology,* 3 (1) (2014), 611-617.

[26] F. Smarandache , Neutrosophy and Neutrosophic Logic , First International Conference on Neutrosophy , Neutrosophic Logic , Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA (2002).

[27] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, *Int. J. Pure Appl. Math.,* 24 (2005), 287-297.

[28] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy Subgroups, *Int. J. Contemp. Math. Sciences,* 6 (8) (2011), 365-372.

[29] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy extension of crisp functions using bridge functions, *Annals of Fuzzy Mathematics and Informatics,* 2 (1) (2011), 1-8.

[30] S. Sebastian and T. V. Ramakrishnan, Multi-Fuzzy Sets, *International Mathematical Forum,* 5 (50) (2010), 2471-2476.

[31] S. Sebastian and Robert John, Multi-fuzzy sets and their correspondence to other sets, *Annals of Fuzzy Mathematics and Informatics,* 11 (2) (2016), 341-348.

[32] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy extensions of functions, *Advance in Adaptive Data Analysis*, 3 (2011), 339-350.

[33] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy topology, *Int. J. Appl. Math.,* 24 (2011), 117-129.

[34] S. Sebastian and T. V.Ramakrishnan, Multi-fuzzy Sets: An Extension of Fuzzy Sets, *Fuzzy Information and Engineering,* Springer, 1 (2011), 35-43.

[35] T. V.Ramakrishnan and S. Sebastian, A study on Multi-fuzzy Sets, *International Journal of Applied Mathematics,* 23 (2010), 713-721.

[36] T. K. Shinoj and S. J. John, Intuitionistic fuzzy multisets and its application in mexdical diagnosis, *World Academy of Science, Engineering and Technology,* 6 (2012), 10-28.

[37] F. Smarandache, n-Valued Refined Neutrosophic Logic and Its Applications in Physics, *Progress in Physics,* 4 (2013), 143-146.

[38] F. Smarandache, A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic, Rehoboth: American Research Press (1998).

[39] A. Syropoulos, On generalized fuzzy multisets and their use in computation, *Iranian Journal Of Fuzzy Systems*, 9 (2) (2012), 113-125.

[40] A. S. Thomas and S. J. John, Multi-fuzzy rough sets and relations, *Annals of Fuzzy Mathematics and Informatics,* x/x (2013), xx-xxx.

[41] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, Single valued neutrosophic sets, *Multispace and Multistructure,* 4 (2010), 410-413.

[42] S. Ye and J. Ye, Dice similarity measure between single valued neutrosophic multisets and its application in medical diagnosis, *Neutrosophic Sets and Systems*, 6 (2014), 48-52.

[43] L. A. Zadeh, Fuzzy Sets, *Inform. and Control,* 8 (1965), 338-353.

Mehmet Şahin
Associate Professor of Mathematics
Department of Mathematics
Gaziantep University
Gaziantep,Turkey
E-mail: mesahin@gantep.edu.tr

Irfan Deli
Assistant Professor of Mathematics
Muallim Rıfat Faculty of Education
Kilis 7 Aralık University
Kilis, Turkey
E-mail: irfandeli@kilis.edu.tr

Vakkas Uluçay
Doctor of Mathematics
Department of Mathematics
Gaziantep University
Gaziantep,Turkey
E-mail: vulucay27@gmail.com