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The Construction of Fraction Gamma Rings and Local Gamma Rings by Using Commutative Gamma Rings

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Abstract. One of the first constructions of algebra is the quotient field of a commutative integral domain, constructed as a set of fractions, which can lead to a very useful technique in commutative ring theory. In this article the researchers considered rings of fractions for gamma rings and some new characterizations were developed in gamma rings of fractions.

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1. Introduction

The notation of a gamma ring was first introduced by Nobusawa [10] as a generalization of a classical ring and afterward Barnes [2] improved the concepts of Nobusawa's Γ-ring and developed the more general Γ-ring in which all classical rings were contained in this Γ-ring. We know, quotient fields are applied for making valuation rings and Dedekind domains and Dedekind domains are used in numbers theory [9].

In this paper the researchers constructed fraction Γ-ring and discussed their characteristics and relations by using local Γ-rings.

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Let R and Γ be two additive abelian groups and there exists a mapping $(x, \gamma, y) \longmapsto x\gamma y$ of $R \times \Gamma \times R \longrightarrow R$, which satisfies the conditions:

$$\begin{aligned} &(i) \quad (x+y)\gamma z = x\gamma z + y\gamma z, \quad x(\gamma_1+\gamma_2)y = x\gamma_1 y + x\gamma_2 y, \\ &\qquad x\gamma(y+z) = x\gamma y + x\gamma z, \\ &(ii) \quad (x\gamma_1 y)\gamma_2 z = x\gamma_1(y\gamma_2 z), \end{aligned}$$

for all $x, y, z \in R$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$. Then R is called a gamma ring.

If there exists $1_R \in R$ and $\gamma_0 \in \Gamma$ such that for all $r \in R$,

$1_R\gamma_0 r = r\gamma_0 1_R = r$, then $1 = 1_R$ is called identity element [13].

Let R be a Γ-ring with 1. An element $a \in R$ is called invertible if there exists $b \in R$ such that $a\gamma_0 b = b\gamma_0 a = 1$, also b is unique and called the multiplicative inverse of a and is denoted by a^{-1}.

An element $a \in R$ is said to be zero -divisor if there exists $b \neq 0$ such that $a\gamma_0 b = b\gamma_0 a = 0$.

Let R be Γ-ring. If for all $a, b \in R$ and for all $\gamma \in \Gamma, a\gamma b = b\gamma a$, then R is called commutative Γ-ring.
A subset I of Γ-ring R is said left(or right) gamma ideal if I is an additive subgroup of R and $R\Gamma I \subseteq I$ $(or I\Gamma R \subseteq I)$ [11].
Let R be a Γ-ring. The ideal generated by $a \in R$ is the intersection of all ideals contain a and
$\langle a \rangle = \{na + x\alpha a + a\beta y + \sum_{i=1}^{k} u_i\gamma_i a\delta_i v_i \mid n, k \in \mathbb{Z}, a, x, y, u_i, v_i \in R,$
$\alpha, \beta, \gamma_i, \delta_i \in \Gamma\}$.
A Γ -ring homomorphism [6] is a mapping f of Γ-ring R to Γ-ring R' such that:
(i) $f(x+y) = f(x) + f(y)$, (ii) $f(x\gamma y) = f(x)\gamma f(y)$, for all $x, y \in R$ and $\gamma \in \Gamma$.
A multiplicatively closed subset of Γ-ring R is a subset S of R such that $1 \in S$ and $s_1\Gamma s_2 \subseteq S$, for all $s_1, s_2 \in S$.

Let R be a Γ-ring with 1 and $* : R \times \Gamma \times R \longrightarrow R$ be a map on R such that $(R - \{0\}, *)$ be a group. Then R is called Γ-field.

We consider the following assumptions

$$\begin{aligned} &(*) \quad x\alpha y\beta z = x\beta y\alpha z, \quad for \ \ all \ \ x, y, z \in R \ \ and \ \ \alpha, \beta \in \Gamma, \\ &(**) \quad (s_1\alpha s_2)\gamma_0(s_1\alpha s_2)\gamma_0(x\beta y) + (s_1\beta s_2)\gamma_0(s_1\beta s_2)\gamma_0(x\alpha y) = 0, \end{aligned}$$

for all $x, y, z \in R, s_1, s_2 \in S, \alpha, \beta \in \Gamma$ [4].

2. Fractions of Gamma Rings

Throughout this section , the word gamma ring R means a commutative gamma ring with 1 and without zero-divisor.

Proposition 2.1. *If a and b are invertible in R , so is $a\gamma_0 b$ and $(a\gamma_0 b)^{-1} = b^{-1}\gamma_0 a^{-1}$.*

Proof.

$$\begin{aligned}(a\gamma_0 b)\gamma_0(b^{-1}\gamma_0 a^{-1}) &= a\gamma_0(b\gamma_0 b^{-1})\gamma_0 a^{-1}\\ &= a\gamma_0 1\gamma_0 a^{-1}\\ &= (a\gamma_0 1)\gamma_0 a^{-1}\\ &= a\gamma_0 a^{-1}\\ &= 1,\end{aligned}$$

and similarly $(b^{-1}\gamma_0 a^{-1})\gamma_0(a\gamma_0 b) = 1$. □

Proposition 2.2. *Let R be a Γ-ring and $S = R - \{0\}$. We define relation $\sim$ on $R \times S$ as follows :*
$(a, s) \sim (b, t) \Longleftrightarrow a\gamma_0 t - b\gamma_0 s = 0$, for $a, b \in R$ and $s, t \in S$. Then $\sim$ is an equivalence relation.

Proof. We show that the relation $\sim$ is reflexive, symmetric and transitive.
Since for all $a \in R$ and $s \in S, a\gamma_0 s - a\gamma_0 s = 0$, so $(a, s) \sim (a, s)$.
If $(a, s) \sim (b, t)$, then $a\gamma_0 t - b\gamma_0 s = 0$, and so $b\gamma_0 s - a\gamma_0 t = 0$. Thus $(b, t) \sim (a, s)$.
If $(a, s) \sim (b, t)$ and $(b, t) \sim (c, u)$, then we have

$$a\gamma_0 t - b\gamma_0 s = 0, \tag{1}$$
$$b\gamma_0 u - c\gamma_0 t = 0. \tag{2}$$

On the other hand, a multiplication by $\gamma_0 u$ of (1) and $\gamma_0 s$ of (2) gives

$$a\gamma_0 t\gamma_0 u - b\gamma_0 s\gamma_0 u = 0, \tag{3}$$
$$b\gamma_0 u\gamma_0 s - c\gamma_0 t\gamma_0 s = 0. \tag{4}$$

Sum of (3) and (4), we obtain

$$a\gamma_0 t\gamma_0 u - c\gamma_0 t\gamma_0 s = 0. \tag{5}$$

By using commutativity, we have

$$(a\gamma_0 u - c\gamma_0 s)\gamma_0 t = 0. \tag{6}$$

We have $t \neq 0$ and R is without zero-divisor, which gives $a\gamma_0 u - c\gamma_0 s = 0$ and thus $(a, s) \sim (c, u)$.
Hence, the proof is complete. □

Theorem 2.3. *Let* $[a, s]$ *denote the equivalence class of* (a, s)*, and* $S^{-1}R$ *denote the set of equivalence classes. If R satisfies the conditions* $(*)$ *and* $(**)$*, we define addition and multiplication of these fractions as follows:*

$$\begin{cases} S^{-1}R \times \Gamma \times S^{-1}R \quad \longrightarrow \quad S^{-1}R \\ [r, s] + [r^{'}, s^{'}] = [r\gamma_0 s^{'} + s\gamma_0 r^{'}, s\gamma_0 s^{'}] \\ [r, s]\gamma[r^{'}, s^{'}] = [r\gamma r^{'}, s\gamma s^{'}], \end{cases}$$

then
(i) these definitions are well-defined.
(ii) $S^{-1}R$ *is a* Γ*-ring with identity element* $[1, 1]$.

Proof. (i): If $[r_1, s_1] = [r_1^{'}, s_1^{'}]$ and $[r_2, s_2] = [r_2^{'}, s_2^{'}]$, then we have

$$r_1\gamma_0 s_1^{'} - s_1\gamma_0 r_1^{'} = 0, \tag{7}$$
$$r_2\gamma_0 s_2^{'} - s_2\gamma_0 r_2^{'} = 0. \tag{8}$$

A multiplication by $s_2\gamma_0 s_2^{'}$ of (7) and $s_1\gamma_0 s_1^{'}$ of (8) gives

$$r_1\gamma_0 s_1^{'}\gamma_0 s_2\gamma_0 s_2^{'} - s_1\gamma_0 r_1^{'}\gamma_0 s_2\gamma_0 s_2^{'} = 0, \tag{9}$$

$$r_2\gamma_0 s_2^{'}\gamma_0 s_1\gamma_0 s_1^{'} - s_2\gamma_0 r_2^{'}\gamma_0 s_1\gamma_0 s_1^{'} = 0. \tag{10}$$

Sum of (9) and (10), we obtain

$$r_1\gamma_0 s_1^{'}\gamma_0 s_2\gamma_0 s_2^{'} - s_1\gamma_0 r_1^{'}\gamma_0 s_2\gamma_0 s_2^{'} + r_2\gamma_0 s_2^{'}\gamma_0 s_1\gamma_0 s_1^{'} - s_2\gamma_0 r_2^{'}\gamma_0 s_1\gamma_0 s_1^{'} = 0.$$

Since R is commutative Γ-ring ,we have

$$r_1\gamma_0 s_2\gamma_0 s_1^{'}\gamma_0 s_2^{'} + r_2\gamma_0 s_1\gamma_0 s_1^{'}\gamma_0 s_2^{'} - r_1^{'}\gamma_0 s_2^{'}\gamma_0 s_1\gamma_0 s_2 - s_1^{'}\gamma_0 r_2^{'}\gamma_0 s_1\gamma_0 s_2 = 0,$$

therefore

$$(r_1\gamma_0 s_2 + r_2\gamma_0 s_1)\gamma_0 s_1^{'}\gamma_0 s_2^{'} - (r_1^{'}\gamma_0 s_2^{'} + s_1^{'}\gamma_0 r_2^{'})\gamma_0 s_1\gamma_0 s_2 = 0, \tag{11}$$

$$[r_1\gamma_0 s_2 + r_2\gamma_0 s_1, s_1\gamma_0 s_2] = [r_1^{'}\gamma_0 s_2^{'} + s_1^{'}\gamma_0 r_2^{'}, s_1^{'}\gamma_0 s_2^{'}], \tag{12}$$
$$[r_1, s_1] + [r_2, s_2] = [r_1^{'}, s_1^{'}] + [r_2^{'}, s_2^{'}]. \tag{13}$$

Thus addition is well-defined.
Now, let $[r_1, s_1] = [r_2, s_2], [r_1^{'}, s_1^{'}] = [r_2^{'}, s_2^{'}]$ and $\gamma = \gamma_1 = \gamma_2$, we have

$$r_1\gamma_0 s_2 - s_1\gamma_0 r_2 = 0, \tag{14}$$
$$r_1^{'}\gamma_0 s_2^{'} - s_1^{'}\gamma_0 r_2^{'} = 0. \tag{15}$$

On the other hand,a multiplication by $r_1^{'}\gamma s_2^{'}$ of (14) and $r_2\gamma s_1$ of (15) gives

$$r_1\gamma_0 s_2\gamma r_1^{'}\gamma s_2^{'} - r_1^{'}\gamma s_2^{'}\gamma s_1\gamma_0 r_2 = 0, \tag{16}$$
$$r_1^{'}\gamma_0 s_2^{'}\gamma r_2\gamma s_1 - s_1^{'}\gamma_0 r_2^{'}\gamma r_2\gamma s_1 = 0. \tag{17}$$

By using the sum of (16) and (17) and applying the condition $(*)$, we obtain

$$r_1\gamma r_1^{'}\gamma_0 s_2\gamma s_2^{'} - s_1\gamma s_1^{'}\gamma_0 r_2\gamma r_2^{'} = 0, \tag{18}$$

or

$$[r_1\gamma r_1^{'}, s_1\gamma s_1^{'}] = [r_2\gamma r_2^{'}, s_2\gamma s_2^{'}], \tag{19}$$

therefore $[r_1, s_1]\gamma[r_1^{'}, s_1^{'}] = [r_2, s_2]\gamma[r_2^{'}, s_2^{'}]$.
Thus the multiplication is well-defined.

Proof (ii). Since S is a multiplicatively closed subset of R and R is $\Gamma - ring$, therefore $r\gamma r^{'} \in R$ and $s\gamma s^{'} \in S$, for all $r, r^{'} \in R, s, s^{'} \in S$ and $\gamma \in \Gamma$. Thus $[r, s]\gamma[r^{'}, s^{'}] = [r\gamma r^{'}, s\gamma s^{'}] \in S^{-1}R$.
For $[r_1, s_1], [r_2, s_2], [r_3, s_3] \in S^{-1}R$ and $\alpha \in \Gamma$, we have

$$([r_1, s_1] + [r_2, s_2])\alpha[r_3, s_3] = [r_1, s_1]\alpha[r_3, s_3] + [r_2, s_2]\alpha[r_3, s_3],$$

because

$$\begin{aligned}([r_1, s_1] + [r_2, s_2])\alpha[r_3, s_3] &= [r_1\gamma_0 s_2 + s_1\gamma_0 r_2, s_1\gamma_0 s_2]\alpha[r_3, s_3] \\ &= [r_1\gamma_0 s_2\alpha r_3 + s_1\gamma_0 r_2\alpha r_3, s_1\gamma_0 s_2\alpha s_3].\end{aligned}$$

Also

$$\begin{aligned}[r_1, s_1]\alpha[r_3, s_3] + [r_2, s_2]\alpha[r_3, s_3] &= [r_1\alpha r_3, s_1\alpha s_3] + [r_2\alpha r_3, s_2\alpha s_3] \\ &= [r_1\alpha r_3\gamma_0 s_2\alpha s_3 + s_1\alpha s_3\gamma_0 r_2\alpha r_3, s_1\alpha s_3\gamma_0 s_2\alpha s_3].\end{aligned}$$

It is easy to see that

$$[r_1\gamma_0 s_2\alpha r_3 + s_1\gamma_0 r_2\alpha r_3, s_1\gamma_0 s_2\alpha s_3] = [r_1\alpha r_3\gamma_0 s_2\alpha s_3 + s_1\alpha s_3\gamma_0 r_2\alpha r_3, s_1\alpha s_3\gamma_0 s_2\alpha s_3]$$

Now, we show that $[r_1, s_1](\alpha+\beta)[r_2, s_2] = [r_1, s_1]\alpha[r_2, s_2] + [r_1, s_1]\beta[r_2, s_2]$, we have

$$\begin{aligned}[r_1, s_1](\alpha+\beta)[r_2, s_2] &= [r_1(\alpha+\beta)r_2, s_1(\alpha+\beta)s_2]\\ &= [r_1\alpha r_2 + r_1\beta r_2, s_1\alpha s_2 + s_1\beta s_2],\\ [r_1, s_1]\alpha[r_2, s_2] + [r_1, s_1]\beta[r_2, s_2] &= [r_1\alpha r_2, s_1\alpha s_2] + [r_1\beta r_2, s_1\beta s_2]\\ &= [(r_1\alpha r_2)\gamma_0(s_1\beta s_2) + (s_1\alpha s_2)\gamma_0 r_1\beta r_2, s_1\alpha s_2\gamma_0 s_1\beta s_2].\end{aligned}$$

We prove that

$$[r_1\alpha r_2 + r_1\beta r_2, s_1\alpha s_2 + s_1\beta s_2] = [r_1\alpha r_2\gamma_0 s_1\beta s_2 + s_1\alpha s_2\gamma_0 r_1\beta r_2, s_1\alpha s_2\gamma_0 s_1\beta s_2],$$

or

$$\begin{aligned}&r_1\alpha r_2\gamma_0 s_1\alpha s_2\gamma_0 s_1\beta s_2 + r_1\beta r_2\gamma_0 s_1\alpha s_2\gamma_0 s_1\beta s_2 - r_1\alpha r_2\gamma_0 s_1\beta s_2\gamma_0 s_1\alpha s_2\\ &- s_1\alpha s_2\gamma_0 r_1\beta r_2\gamma_0 s_1\alpha s_2 - r_1\alpha r_2\gamma_0 s_1\beta s_2\gamma_0 s_1\beta s_2 - s_1\alpha s_2\gamma_0 r_1\beta r_2\gamma_0 s_1\beta s_2 = 0.\end{aligned}$$

But by using commutativity and the condition $(**)$, the above relation is satisfied.
Also we have

$$\begin{aligned}([r_1, s_1]\alpha[r_2, s_2])\beta[r_3, s_3] &= [r_1\alpha r_2, s_1\alpha s_2]\beta[r_3, s_3]\\ &= [(r_1\alpha r_2)\beta r_3, (s_1\alpha s_2)\beta s_3]\\ &= [r_1\alpha(r_2\beta r_3), s_1\alpha(s_2\beta s_2)]\\ &= [r_1, s_1]\alpha[r_2\beta r_3, s_2\beta s_3]\\ &= [r_1, s_1]\alpha([r_2, s_2]\beta[r_3, s_3]).\end{aligned}$$

For all $[r, s] \in S^{-1}R$, we have

$$[r, s]\gamma_0[1, 1] = [r\gamma_0 1, s\gamma_0 1] = [r, s],$$

and similarly since $[1, 1]\gamma_0[r, s] = [r, s]$, thus $[1, 1] \in S^{-1}R$ is an identity element, the proof is complete. $\square$

The Γ -ring $S^{-1}R$ is called the Γ-ring of fraction of R with respect to S.

Proposition 2.4. *Let R, S be in Proposition 2.2. Then*
(i) $[0, s] = [0, 1]$, *for all* $s \in S$.
(ii) $[r, s] = [r\gamma r', s\gamma r']$, *for all* $r, r' \in R$, $s \in S$ *and* $\gamma \in \Gamma$.
(iii) $-(x\alpha y) = x(-\alpha)y$, *for all* $x, y \in R$, $\alpha \in \Gamma$.
(iv) $[r, r] = [1, 1]$, *for all* $r \in R$.

Proof. (i) Since $0\gamma_0 s - 1\gamma_0 0 = 0, \quad so \quad [0,1] = [0,s]$.
(ii) Since R is commutative Γ-ring, then $r\gamma_0 s\gamma r^{'} - s\gamma_0 r\gamma r^{'} = 0$ and therefore

$$[r,s] = [r\gamma r^{'}, s\gamma r^{'}]. \tag{20}$$

(iii) We have $x(-\alpha)y + x(\alpha)y = x(-\alpha+\alpha)y = 0$, thus $-(x\alpha y) = x(-\alpha)y$.
(iv) Since $r\gamma_0 1 - 1\gamma_0 r = r - r = 0$, so $[r,r] = [1,1]$, for all $r \in R$. □

Theorem 2.5. *If R is a Γ-ring and $S = R - \{0\}$, then $S^{-1}R$ is a Γ-field.*

Proof. By using Theorem 2.1, $(S^{-1}R, +, \cdot)$ is a Γ-ring with identity element $[1,1]$, thus for every $r, s \in S$, we prove that $[r,s]^{-1} = [s,r]$. By using commutativity Γ-ring R and Proposition 2.3 (iv), we have

$$[r,s]\gamma_0[s,r] = [r\gamma_0 s, s\gamma_0 r] = [r\gamma_0 s, r\gamma_0 s] = [1,1].$$

Similarly $[s,r]\gamma_0[r,s] = [1,1]$.
Now, we prove that $(S^{-1}R, \cdot)$ is associative. Since R is a Γ-ring, we have

$$\begin{aligned}([r_1,s_1]\gamma_1[r_2,s_2])\gamma_2[r_3,s_3] &= [r_1\gamma_1 r_2, s_1\gamma_1 s_2]\gamma_2[r_3,s_3] \\ &= [(r_1\gamma_1 r_2)\gamma_2 r_3, (s_1\gamma_1 s_2)\gamma_2 s_3] \\ &= [r_1\gamma_1(r_2\gamma_2 s_3), s_1\gamma_1(s_2\gamma_2 s_3)] \\ &= [r_1,s_1]\gamma_1([r_2,s_2]\gamma_2[r_3,s_3]).\end{aligned}$$

At the end, we prove that $(S^{-1}R, \cdot)$ is commutative. Since R is a commutative Γ-ring, for every $\gamma \in \Gamma, r_1, r_2 \in R, s_1, s_2 \in S$ we have

$$\begin{aligned}[r_1,s_1]\gamma_1[r_2,s_2] &= [r_1\gamma r_2, s_1\gamma s_2] \\ &= [r_2\gamma r_1, s_2\gamma s_1] \\ &= [r_2,s_2]\gamma[r_1,s_1].\end{aligned}$$

Hence $(S^{-1}R, +, \cdot)$ is a Γ-field. □

At the end of this section, we give an example of matrices that are not rings under addition and matrix multiplication ,but we will make a gamma ring of them.

Example 2.6. Let $\mathbb{Z}$ be integers rings and $M_{m\times n}(\mathbb{Z})$ be the set of all $m \times n$ matrices with entries in $\mathbb{Z}$. We consider
$R = \{\begin{bmatrix} x & x \end{bmatrix} | x \in \mathbb{Z}\} \subseteq M_{1\times 2}$ and $\Gamma = \{\begin{bmatrix} n \\ o \end{bmatrix} | n \in \mathbb{Z}\} \subseteq M_{2\times 1}$.
and we define

$$\begin{cases} . : R \times \Gamma \times R \longrightarrow R \\ \begin{bmatrix} x & x \end{bmatrix} . \begin{bmatrix} n \\ o \end{bmatrix} . \begin{bmatrix} y & y \end{bmatrix} = \begin{bmatrix} nxy & nxy \end{bmatrix} \end{cases}$$

,for all $\begin{bmatrix} x & x \end{bmatrix}$, $\begin{bmatrix} y & y \end{bmatrix}$ in R and for all $\begin{bmatrix} n \\ o \end{bmatrix}$ in Γ.
It is easy to see that R is a Γ-ring. We show that R is integral domain with $1_R = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\gamma_0 = \begin{bmatrix} 1 \\ o \end{bmatrix}$.
Hence, if we consider $S = R - \{0\}$, then by using Theorem 2.2, $S^{-1}R$ is a Γ-field.

Proof. For $\begin{bmatrix} x & x \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$, then $x \neq 0$ and if $\begin{bmatrix} y & y \end{bmatrix} \in R$, we have

$$\begin{aligned} \begin{bmatrix} x & x \end{bmatrix} . \begin{bmatrix} 1 \\ o \end{bmatrix} . \begin{bmatrix} y & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} &\Rightarrow \begin{bmatrix} xy & xy \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \\ &\Rightarrow xy = 0 \\ &\Rightarrow y = 0, \end{aligned}$$

then $\begin{bmatrix} y & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

Also for all $\begin{bmatrix} x & x \end{bmatrix} \in R$ we have
$\begin{bmatrix} 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 \\ o \end{bmatrix} . \begin{bmatrix} x & x \end{bmatrix} = 1. \begin{bmatrix} x & x \end{bmatrix} = \begin{bmatrix} x & x \end{bmatrix}$ and
$\begin{bmatrix} x & x \end{bmatrix} . \begin{bmatrix} 1 \\ o \end{bmatrix} . \begin{bmatrix} 1 & 1 \end{bmatrix} = x. \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} x & x \end{bmatrix}$, hence R has identity element.

With simple calculations, we get the equivalence class of $[\begin{bmatrix} x & x \end{bmatrix}, \begin{bmatrix} y & y \end{bmatrix}]$ is $\{ [\begin{bmatrix} z & z \end{bmatrix}, \begin{bmatrix} t & t \end{bmatrix}] \mid xt = yz, x, z \in \mathbb{Z}, y, t \in \mathbb{Z} - \{0\}\}$. $\square$

3. Homomorphisms of Gamma Rings

In this section, the notion homomorphism of gamma rings is defined and some theorems will be proved.

Theorem 3.1. Let R be a Γ-ring and $S^{-1}R$ be a Γ-ring of fraction in Theorem 2.1. Then the mapping $f : R \longrightarrow S^{-1}R$ such that $f(r) = [r, 1]$ is a Γ-ring homomorphism.

Proof. At first we show that f is well-defined. If $r_1 = r_2$, then $r_1 - r_2 = 0$. Since $r_1 = r_1\gamma_0 1$ and $r_2 = r_2\gamma_0 1$, then $r_1\gamma_0 1 - r_2\gamma_0 1 = 0$ and so $[r_1, 1] = [r_2, 1]$.
Now we prove that $f(r_1 + r_2) = f(r_1) + f(r_2)$ and $f(r_1\gamma r_2) = f(r_1)\gamma f(r_2)$, for all $r_1, r_2 \in R$ and $\gamma \in \Gamma$. We have

$$f(r_1) + f(r_2) = [r_1, 1] + [r_2, 1] = [r_1\gamma_0 1 + r_2\gamma_0 1, 1\gamma_0 1] = [r_1 + r_2, 1] = f(r_1 + r_2)$$

We have $f(r_1)\gamma f(r_2) = [r_1, 1]\gamma[r_2, 1] = [r_1\gamma r_2, 1\gamma 1]$, but $f(r_1\gamma r_2) = [r_1\gamma r_2, 1]$, we get that

$$[r_1\gamma r_2, 1] = [r_1\gamma r_2, 1\gamma 1] \quad \Leftrightarrow \quad r_1\gamma r_2\gamma_0 1\gamma 1 - r_1\gamma r_2\gamma_0 1 = 0$$
$$\Leftrightarrow \quad r_1\gamma r_2\gamma 1 - r_1\gamma r_2 = 0.$$

If we put $\alpha = -\gamma$, $\beta = \gamma_0$, $x = r_1, y = r_2$ and $s_1 = s_2 = 1$, in condition $(**)$, we have

$$1(-\gamma)1\gamma_0 1(-\gamma)1\gamma_0 r_1\gamma_0 r_2 + r_1(-\gamma)r_2\gamma_0 1\gamma_0 1\gamma_0 1\gamma_0 1 = 0.$$

By Proposition 2.3 (iii), we obtain

$$1\gamma 1 r_1\gamma_0 r_2 - r_1\gamma r_2 = 0.$$

Also by the condition $(*)$, we have

$$1\gamma 1\gamma_0 r_1\gamma r_2 - r_1\gamma r_2 = 0.$$

Since $1\gamma_0 r_1 = r_1$, we get that

$$1\gamma r_1\gamma r_2 - r_1\gamma r_2 = 0.$$

Hence the theorem is proved. □

Proposition 3.2. Let R and $R^{'}$ be Γ-rings with identity elements and $f : R \longrightarrow R^{'}$ be a Γ-ring epimorphism. Then $f(1_R) = 1_{R'}$.

Proof. We prove that $f(1_R)\gamma_0 r^{'} = r^{'}\gamma_0 f(1_R) = r^{'}$, for all $r^{'} \in R^{'}$. Since f is surjective , there exists $r \in R$ such that $f(r) = r^{'}$. We have

$$f(1_R)\gamma_0 r^{'} = f(1_R)\gamma_0 f(r) = f(1_R\gamma_0 r) = f(r) = r^{'},$$
$$r^{'}\gamma_0 f(1_R) = f(r)\gamma_0 f(1_R) = f(r\gamma_0 1_R) = f(r) = r^{'}.$$

Hence $f(1_R) = 1_{R'}$. □

Proposition 3.3. If R and $R^{'}$ are Γ-rings with identity elements without zero-divisor and $f : R \longrightarrow R^{'}$ is a non-zero Γ-ring homomorphism. Then $f(1_R) = 1_{R'}$.

Proof. We have

$$f(1_R) = f(1_R\gamma_0 1_R) = f(1_R)\gamma_0 f(1_R)$$
$$\Rightarrow \quad f(1_R) - f(1_R)\gamma_0 f(1_R) = 0$$
$$\Rightarrow \quad f(1_R)\gamma_0(1_{R'} - f(1_R)) = 0$$
$$\Rightarrow \quad 1_{R'} - f(1_R) = 0.$$

Hence $f(1_R) = 1_{R'}$. $\square$

Proposition 3.4. Let R and $R^{'}$ be Γ-rings with identity elements, without zero-divisor and $f : R \longrightarrow R^{'}$ is a non-zero Γ-ring homomorphism. Then $f(a^{-1}) = (f(a))^{-1}$.

Proof. Suppose $a \in R$ is invertible and a^{-1} is inverse of a. We have

$$\begin{aligned} f(a\gamma_0 a^{-1}) = f(1_R) = f(a^{-1}\gamma_0 a) & \\ \Rightarrow \quad & f(a)\gamma_0 f(a^{-1}) = 1_{R'} = f(a^{-1})\gamma_0 f(a) \\ \Rightarrow \quad & f(a^{-1}) = (f(a))^{-1}. \quad \square \end{aligned}$$

4. Local Gamma Rings

In this section, local gamma rings is defined and will be given several conditions equivalent for local gamma rings.

Definition 4.1. *A Γ-ideal P in Γ-ring R is prime [15], if $P \neq R$ and if $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$, for every Γ-ideals A and B in R.*

Theorem 4.2. *If R is a commutative Γ- ring and P is a Γ-ideal such that $P \neq R$ and $a\gamma b \in P$, for $a, b \in R$ and all $\gamma \in \Gamma$ it implies that $a \in P$ or $b \in P$, then P is prime and conversely.*

Proof. $\Longrightarrow$) If A and B are gamma ideals in R such that $A\Gamma B \subseteq P$, but $A \nsubseteq P$ and $B \nsubseteq P$, then there are $a_0 \in A$ and $b_0 \in B$ such that a_0 and b_0 are not in P.
Since $A\Gamma B \subseteq P$, then for every $\gamma \in \Gamma$, $a_0\gamma b_0 \in A\Gamma B \subseteq P$ and by assumption $a_0 \in P$ or $b_0 \in P$, this is a contradiction. Thus $A \subseteq P$ or $B \subseteq P$.
$\Longleftarrow$) Let P be a prime gamma ideal and $a\gamma b \in P$ for every $a, b \in R$ and for all $\gamma \in \Gamma$, then $\langle a\rangle\Gamma\langle b\rangle \subseteq P$ and therefore $\langle a\rangle \subseteq P$ or $\langle b\rangle \subseteq P$, but $a \in \langle a\rangle$ and $b \in \langle b\rangle$ thus $a \in P$ or $b \in P$. $\square$

Theorem 4.3. *In a commutative Γ-ring R with identity, an ideal P is prime if and only if $S = R - P$ is multiplicatively closed subset.*

Proof. $\Longrightarrow$) Let gamma ideal P be prime in R and $s_1, s_2 \in S$. Then s_1 and s_2 aren't in P ($S = R - P$). Since P is prime, for every $\gamma \in \Gamma$, $s_1\gamma s_2$ isn't in P, hence for all $\gamma \in \Gamma$, $s_1\gamma s_2 \in S$, therefore $s_1\Gamma s_2 \subseteq S$.
$\Longleftarrow$) Suppose $S = R - P$ is a multiplicatively closed subset in R, then $1 \in S$ and so $S \neq \emptyset$, i.e $P \neq R$.

If $a\gamma b \in P$ for every $a, b \in R$ and for every $\gamma \in \Gamma$, then $a\gamma b$ isn't in S. Since S is multiplicatively closed subset, then a or b aren't in S, i.e $a \in P$ or $b \in P$. $\square$

Notation

Let gamma ideal P be prime in R and $S = R - P$. Then we write $A_{\Gamma P} = S^{-1}R$.

Theorem 4.4. *In a commutative* Γ*-ring* R *with identity, if gamma ideal* P *is prime and* $S = R - P$*, then the set* $M = \{[a, s] |\ a \in P,\ s \in S\}$ *is an ideal of* $A_{\Gamma P}$*.*

Proof. Since $0 \in P$, then $[0, s] \in M$, for $s \in S$ and so $M \neq \emptyset$. To show that $(M, +)$ is subgroup, for every $a, b \in P$ and $s, s^{'} \in S$, we have $a\gamma_0 s^{'}$ and $b\gamma_0 s \in P$ (P is an Γ-ideal) and $s\gamma_0 s^{'} \in S$ (S is a multiplicatively closed subset). Thus $[a, s] - [b, s^{'}] = [a, s] + [-b, s^{'}] = [a\gamma_0 s^{'} - b\gamma_0 s, s\gamma_0 s^{'}] \in M$.
To show that $M\Gamma A_{\Gamma P} \subseteq M$, we consider $[a, s]\gamma[b, s^{'}] \in M\Gamma A_{\Gamma P}$.
Since P is an ideal in R and S is a multiplicatively closed subset, then $a\gamma b \in P$ and $s\gamma s^{'} \in S$. Thus $[a, s]\gamma[b, s^{'}] = [a\gamma b, s\gamma s^{'}] \in M$, i.e $M\Gamma A_{\Gamma P} \subseteq M$. $\square$

Theorem 4.5. *Let* M *be the set of all non-invertible elements of* Γ*-ring* R*, then the following properties are equivalent:*
(1) *M is additively closed* ($\forall a_1, a_2 \in M, a_1 + a_2 \in M$),
(2) *M is a two-sided gamma ideal of R,*
($3r$) *M is the largest proper right gamma ideal,*
($3l$) *M is the largest proper left gamma ideal,*
($4r$) *In gamma ring R there exists a largest proper right ideal,*
($4l$) *In gamma ring R there exists a largest proper left ideal,*
($5r$) *For every $r \in R$ either r or $1 - r$ is right invertible,*
($5l$) *For every $r \in R$ either r or $1 - r$ is left invertible,*
(6) *For every $r \in R$ either r or $1 - r$ is invertible.*

Proof. (1) $\Rightarrow$ (2): Let M be additively closed. At first, we show that every right (left) invertible element is invertible. If $b \in R$ is right invertible, then there exists $b^{'} \in R$ such that $b\gamma_0 b^{'} = 1$, to show that $b^{'}\gamma_0 b = 1$, we have two cases.
Case 1. If $b^{'}\gamma_0 b$ isn't in M, then there is $s \in R$ with $s\gamma_0(b^{'}\gamma_0 b) = 1$. A right multiplication by $\gamma_0 b^{'}$ gives

$$\begin{aligned} s\gamma_0 b^{'}\gamma_0 b\gamma_0 b^{'} = 1\gamma_0 b^{'} &\implies s\gamma_0 b^{'}\gamma_0 1 = b^{'} \\ &\implies s\gamma_0 b^{'} = b^{'} \\ &\implies b^{'}\gamma_0 b = 1. \end{aligned}$$

Case 2. If $b^{'}\gamma_0 b \in M$, then $1 - b^{'}\gamma_0 b$ isn't in M, otherwise if $1 - b^{'}\gamma_0 b \in M$, we have $b^{'}\gamma_0 b \in M$ and M is an additively closed set, then

$$1 = (1 - b^{'}\gamma_0 b) + b^{'}\gamma_0 b \in M.$$

It is a contradiction.
Thus there exists $s \in R$ such that $s\gamma_0(1 - b^{'}\gamma_0 b) = 1$. The right multiplication by $\gamma_0 b^{'}$ gives

$$\begin{aligned} s\gamma_0(1 - b^{'}\gamma_0 b)\gamma_0 b^{'} = 1\gamma_0 b^{'} &\Longrightarrow s\gamma_0(1\gamma_0 b^{'} - b^{'}\gamma_0 b\gamma_0 b^{'}) = b^{'} \\ &\Longrightarrow s\gamma_0(b^{'} - b^{'}\gamma_0 1) = b^{'} \\ &\Longrightarrow s\gamma_0(b^{'} - b^{'}) = b^{'} \\ &\Longrightarrow 0 = b^{'}, \end{aligned}$$

it is contradiction to $b\gamma_0 b^{'} = 1$. Hence by using case 1, b is invertible.
Now, we prove that for every $m \in M$, $r \in R$ and $\gamma \in \Gamma$, $r\gamma m \in M$ and $m\gamma r \in M$.
Suppose $r\gamma m$ is not in M, then there exists $s \in R$ such that $r\gamma m\gamma_0 s = 1$. By using case 1, $s\gamma_0 r\gamma m = 1$ and by the contradiction $(*)$, $s\gamma r\gamma_0 m = 1$. Thus $s\gamma r$ is inverse of m, in contradiction with $m \in M$. Hence $r\gamma m \in M$ and similarly $m\gamma r \in M$.
Let $\sum_{i=1}^{n} r_i\gamma_i m_i \in R\Gamma M$. Since $r_i\gamma_i m_i \in M$, for every $1 \leqslant i \leqslant n$ and M is an additively closed set, then $\sum_{i=1}^{n} r_i\gamma_i m_i \in M$, i.e $R\Gamma M \subseteq M$ and similarly $M\Gamma R \subseteq M$. Hence M is two-sided gamma ideal of R.
(2) $\Longrightarrow$ ($3r$): Let M be two-sided gamma ideal in R. Then M is right gamma ideal. Since 1 isn't in M, then $M \neq R$.
Let B be proper right gamma ideal in R. We show that $B \subseteq M$. If $b \in B$, then $b\Gamma R$ is right gamma ideal of B and therefore $b\Gamma R$ is a proper right gamma ideal in R. Thus b isn't invertible and hence $b \in M$, i.e $B \subseteq M$.
($3r$) $\Longrightarrow$ ($4r$): It is clearly that M is a largest proper right gamma ideal.
($4r$) $\Longrightarrow$ ($5r$). Let N be the largest proper right ideal. Let $r \in R$ and r and $1 - r$ aren't invertible. Then $r\Gamma R$ and $(1 - r)\Gamma R$ are proper gamma ideals of R, hence $r\Gamma R \subseteq N$ and $(1 - r)\Gamma R \subseteq N$.
We have $1 = (1 - r)\gamma_0 1 + r\gamma_0 1 \in (1 - r)\Gamma R + r\Gamma R \subseteq N$, i.e $1 \in N$, in contradiction with $N \subsetneqq R$.
($5r$) $\Longrightarrow$ (6): It suffices to show that every right invertible element is invertible.
Let b has right inverse like $b^{'}$. Then $b\gamma_0 b^{'} = 1$.
Let $b^{'}\gamma_0 b \in R$. We have two cases:
Case 1. $b^{'}\gamma_0 b$ is right invertible, hence there is $s \in R$ such that $b^{'}\gamma_0 b\gamma_0 s = 1$.

The left multiplication by $(b\gamma_0)$ gives

$$\begin{aligned} b\gamma_0 b^{'}\gamma_0 b\gamma_0 s = b\gamma_0 1 &\Longrightarrow 1\gamma_0 b\gamma_0 s = b \\ &\Longrightarrow b\gamma_0 s = b \\ &\Longrightarrow b^{'}\gamma_0 b = 1. \end{aligned}$$

Case 2. $(1 - b^{'}\gamma_0 b)$ is right invertible, hence there is $s \in R$ with $(1 - b^{'}\gamma_0 b)\gamma_0 s = 1$. The left multiplication by $(b\gamma_0)$ gives

$$\begin{aligned} b\gamma_0(1 - b^{'}\gamma_0 b)\gamma_0 s = b\gamma_0 1 &\Longrightarrow (b\gamma_0 1 - b\gamma_0 b^{'}\gamma_0 b)\gamma_0 s = b \\ &\Longrightarrow (b - 1\gamma_0 b)\gamma_0 s = b \\ &\Longrightarrow (b - b)\gamma_0 s = b \\ &\Longrightarrow 0 = b. \end{aligned}$$

It is in contradiction to $b\gamma_0 b^{'} = 1$. Hence by using case 1, $b^{'}\gamma_0 b = 1$.
(6) $\Longrightarrow$ (1). Suppose $m_1, m_2 \in M$, we show that $m_1 + m_2 \in M$.
If $m_1 + m_2$ isn't in M, then $m_1 + m_2$ is invertible, so there is $s \in R$ with $(m_1 + m_2)\gamma_0 s = 1$ thus $m_1\gamma_0 s = 1 - m_2\gamma_0 s$.
But $m_1\gamma_0 s \in M$ must be held, otherwise $m_1\gamma_0 s$ is invertible, i.e there is $b \in R$ such that $m_1\gamma_0 s\gamma_0 b = 1$ and then $s\gamma_0 b$ is right inverse of m_1. Since (6) $\Longrightarrow$ $(5r)$ holds, we can use the fact that every right invertible element is invertible. Hence m_1 isn't in M, there is contradiction.
Similarly it is proved that $m_2\gamma_0 s \in M$ and by using (6), $(1-m_2\gamma_0 s)$ is invertible and therefore $m_1\gamma_0 s$ is invertible, in contradiction with $m_1\gamma_0 s \in M$.

Definition 4.6. *A gamma ring R which satisfies the equivalent properties of Theorem 4.4 is called local gamma ring.*

Corollary 4.7. *$A_{\Gamma P}$ is a local Γ-ring.*

Proof. It follows from Theorem 4.3 and Theorem 4.4. □

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