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Mizoguchi-Takahashi's Fixed Point Theorem Concerning $\tau-$Distance

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Abstract. In this paper by using the notion of τ-distance, we will prove Mizoguchi-Takahashi's fixed point theorem, which is a generalization of fixed point theorem which has been given by Nadler.

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1. Introduction

In 1922, Banach ([1]) proved the famous fixed point theorem known as the Banach contraction principle which is a very useful tool in nonlinear analysis, control theory, economic theory and global analysis. Later on, the theorem has been generalized in several directions. For example, In 1969, Nadler generalized it to set-valued mappings and proved some fixed point theorems about set-valued contraction mappings ([6]).
A point x is said to be a fixed point of a single-valued mapping f(set-valued mapping F) provided $f(x) = x, (x \in F(x))$. We denote by $CB(X)$ the class of all nonempty bounded closed subset of X. Since the mapping $i : X \to CB(X)$, given by $i(x) = \{x\}$ for each $x \in X$ is an isometry, the fixed point theorem in this paper for set-valued mapping are generalized of their single-valued analogues.

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Definition 1.1. *Let (X, d) be a metric space. The Hausdorff metric with respect to d, denoted by H and defined by*

$$H(A, B) = \max\{sup_{u\in A} d(u, B), sup_{\nu\in B} d(\nu, A)\}$$

for every $A, B \in CB(X)$, where $d(x, A) = inf\{d(x, y); y \in A\}$ for every $A \subset X$.

Theorem 1.2. *(Nadler [6]), Let (X, d) be a complete metric space and T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $H(Tx, Ty) \leqslant rd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.*

Mizoguchi and Takahashi ([5]) proved a generalization of the theorem, which is a partial answer of problem 9 in Rich ([7]). See also [3,4,8].

Theorem 1.3. *(Mizoguchi and Takahashi ([5])), Let (X, d) be a complete metric space and T be a mapping from X into $CB(X)$. Assume $H(Tx, Ty) \leqslant \alpha(d(x, y)).d(x, y)$ for all $x, y \in X$, where α is a function from $[0, \infty)$ into $[0, 1)$ satisfying* $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then there exists $z_0 \in X$ such that $z_0 \in Tz_0$.

Recently, Suzuki ([9]) introduced the notion of τ-distance on a metric space, which is the generalization of the concept of ω-distance and Tatarou's distance. He also improved some fixed point theorems. In this paper, by using the notion of τ-distance, we will prove Mizoguchi-Takahashi's fixed point theorem for set-valued mappings which is a real generalization of Nadler's.

2. τ-Distance

Through out of this paper, we denote by N, the set of all positive integers and by R_+, the set of all nonnegative real numbers. In this section, we state the definition of τ-distance which was first introduce by Suzuki [9]. Then we give some propertises will be connected to τ-distance.

Definition 2.1. ([9]). *Let (X, d) be a metric space. A function p from $X \times X$ into $[0, \infty)$ is called a τ-distance on X, if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and following are satisfied;*

(τ_1) *$p(x, z) \leqslant p(x, y) + p(y, z)$ for all $x, y, z \in X$;*

(τ_2) *$\eta(x, 0) = 0$ and $\eta(x, t) \geqslant t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable;*

(τ_3) *$lim_n x_n = x$ and $lim_n sup\{\eta(z_n, p(z_n, x_m)) : m \geqslant n\} = 0$ imply $p(w, x) \leqslant lim_n inf p(w, x_n)$ for all $w \in X$;*

(τ_4) *$lim_n sup\{p(x_n, y_m) : m \geqslant n\} = 0$ and $lim_n \eta(x_n, t_n) = 0$ imply $lim_n \eta(y_n, t_n) = 0$;*

(τ_5) *$lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $lim_n d(x_n, y_n) = 0$.*

We may replace (τ_2) by $(\tau_2)'$ (see [9]):

$(\tau_2)'$ $inf\{\eta(x, t) : t > 0\}$ for all $x \in X$, and η is nondecreasing in its second variable.

Many useful examples of τ-distance are stated in [9,10].

Definition 2.2. ([9]). *Let (X, d) be a metric space and p a τ-distance on X. A sequence $\{x_n\}$ of X is called p-Cauchy if there exist a function η from $X \times R_+$ into R_+satisfying $(\tau_2) - (\tau_5)$ and a sequence $\{z_n\}$ of X such that $lim_n sup\{\eta(z_n, p(z_n, x_m)) : m \geqslant n\} = 0$.*

The following two lemmas are very useful in the proof of fixed point theorem in Section 3.

Lemma 2.3. ([9]). *Let (X, d) be a metric space and a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying $lim_n sup\{p(x_n, y_m) : m \geqslant n\} = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $lim_n d(x_n, y_n) = 0$.*

Lemma 2.4. ([9]). *Let (X, d) be a metric space and p a τ-distance on X. If a sequence $\{x_n\}$ in X satisfies $lim_n sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a p-Cauchy sequence. Moreover, if a sequence $\{y_n\}$ in X satisfies $lim_n p(x_n, y_n) = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $lim_n d(x_n, y_n) = 0$.*

3. Main Result

In this section by using τ-distance, we give a proof of Mizoguchi-Takahashi's fixed point theorem which is simpler than previous proofs ([2,5]).

Theorem 3.1. *Let (X,d) be a complete metric space and let p be a τ-distance on X. Let T be a mapping from X into $CB(X)$. Assume that $H(Tx,Ty) \leqslant \alpha(p(x,y)).p(x,y)$ for all $x,y \in X$, where α is a function from $[0,\infty)$ into $[0,1)$ satisfying $\limsup\limits_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,\infty)$. Then there exists $z_0 \in X$ such that $z_0 \in Tz_0$ and $p(z_0,z_0) = 0$.*

Proof. Define a function β from $[0,\infty)$ into $[0,1)$ by $\beta(t) = (\alpha(t)+1)/2$. Then the following hold:

1) $\limsup\limits_{s\to t^+} \beta(s) < 1$.

2) $\forall x,y \in X \quad , \quad u \in Tx \quad \exists \nu \in Ty \; ; \; p(u,\nu) \leqslant \beta(p(x,y)).p(x,y)$.

Fix $u_0 \in X$ and $u_1 \in Tu_0$. Then there exists $u_2 \in Tu_1$ such that $p(u_1,u_2) \leqslant \beta(p(u_0,u_1)).p(u_0,u_1)$. Thus, we have a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$p(u_{n+1},u_{n+2}) \leqslant \beta(p(u_n,u_{n+1})).p(u_n,u_{n+1})$$

for all $n \in N$. Since $\beta(t) < 1$ for all $t \in [0,\infty)$ *then* $\{p(u_n,u_{n+1})\}$ is a nonincreasing sequence. Hence $\{p(u_n,u_{n+1})\}$ converges to some nonnegative real number λ. Since $\limsup\limits_{s\to \lambda^+} \beta(s) < 1$ and $\beta(\lambda) < 1$, there exists $r \in [0,1)$ and $\varepsilon > 0$ such that $\beta(s) \leqslant r$ for all $s \in [\lambda, \lambda+\varepsilon)$. We can take $k \in N$ such that $\lambda \leqslant p(u_n,u_{n+1}) < \lambda+\varepsilon$ for all $n \in N$ with $n \geqslant k$. Since

$$p(u_{n+2},u_{n+1}) \leqslant \beta(p(u_n,u_{n+1})).p(u_n,u_{n+1}) < r.p(u_n,u_{n+1}), \tag{1}$$

for any $n \in N$ with $n \geqslant k$, then we have

$$\begin{aligned} \textstyle\sum_{n=1}^{\infty} p(u_n,u_{n+1}) &= \textstyle\sum_{n=1}^{k} p(u_n,u_{n+1}) + \sum_{n=k+1}^{\infty} p(u_n,u_{n+1}) \\ &\leqslant \textstyle\sum_{n=1}^{k} p(u_n,u_{n+1}) + \sum_{n=1}^{\infty} r^n.p(u_k,u_{k+1}) < \infty, \end{aligned}$$

hence $lim_n sup\ p(u_n,u_{n+1}) = 0$. By Lemma 2.4., $\{u_n\}$ is a p-Cauchy sequence and hence, by Lemma 2.3., $\{u_n\}$ is a Cauchy sequence. Since

X is complete, $\{u_n\}$ converges to some point $\nu_0 \in X$. From (τ_3), we have

$$p(u_n, \nu_0) \leqslant lim_m inf p(u_n, u_m) \leqslant \frac{r^n}{1-r} p(u_k, u_{k+1}) \tag{2}$$

for $m > n > k$. We have also $w_n \in T\nu$. Since

$$lim_n sup\{p(u_n, w_m); m \geqslant n\} = 0$$

by Lemma 2.3., $\{w_n\}$ is a p-Cauchy sequence and we have $lim_n d(u_n, w_n) = 0$. But

$$d(w_n, \nu_0) \leqslant d(w_n, u_n) + d(u_n, \nu_0)$$

then $\{w_n\}$ converges to ν_o. By closedness of $T\nu_0$, we have $\nu_0 \in T\nu_0$. So we have a sequence $\{\nu_n\}$ such that $\nu_{n+1} \in T\nu_n$ and by (1), for all $n \in N$

$$p(\nu_0, \nu_{n+1}) \leqslant \beta(p(\nu_0, \nu_n)).p(\nu_0, \nu_n) \leqslant r.p(\nu_0, \nu_n).$$

Also

$$p(\nu_0, \nu_n) \leqslant r.p(\nu_0, \nu_{n-1}) \leqslant ... \leqslant r^n.p(\nu_0, \nu_0).$$

Hence

$$lim_n sup\ p(u_n, \nu_n) \leqslant lim_n(p(u_n, \nu_0) + p(\nu_0, \nu_n)) = 0.$$

By Lemma 2.4., $\{\nu_n\}$ is a p-Cauchy sequence and converges to ν_0. So we have

$$p(\nu_0, \nu_0) \leqslant lim_n p(\nu_0, \nu_n) = 0.$$

This completes the proof. □

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