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# Weak Shadowing as a Generic Property for IFS's

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**Abstract.** In this paper we consider shadowing and weak shadowing properties for iterated function systems *IFS* and give some results on these concepts. At first, a sufficient condition for shadowing property is given and by this result we present two IFS which have the shadowing property. It is proved that every uniformly expanding as well as every uniformly contracting IFS has the weak shadowing property. By an example we show that in IFS's shadowing property does not imply weak shadowing property. Finally we have the main result of the paper and prove that the weak shadowing is a generic property in the set of all IFS's.

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## 1. Introduction

The notion of shadowing is an important tool for studying properties of discrete dynamical systems. From numerical point of view, if a dynamical system has the shadowing property, then numerically obtained orbits reflect the real behavior of trajectories of the systems(see [5, 15, 18]). The so-called *weak shadowing property* is introduced and studied in [3, 12] and this is proved that shadowing and weak shadowing are

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$C^0$- generic properties in $H(X)$, where $H(X)$ is the set of all homeomorphisms of a compact topological space $X$. Specially, the space $X$ is one of the following:
(i) a topological manifold with boundary ($dim(X) \geqslant 2$ if $\partial X \neq \emptyset$.),
(ii) a Cartesian product of a countably infinite number of manifolds with nonempty boundary,
(iii) a Cantor set,
then weak shadowing is a generic property in $H(X)$ [12].
It was proved that every diffeomorphism on a smooth closed surface having the weak shadowing property is O-stable (for low dimensions), but is not structurally stable in general [16, 19, 20]. In [4] the authors prove that on an oriented smooth closed surface, weak shadowing property and structural stability are equivalent. Iterated function systems( **IFS**), are used for the construction of deterministic fractals and have found numerous applications, in particular to image compression and image processing [1]. Important notions in dynamics like attractors, minimality, transitivity, and shadowing can be extended to IFS (see [2, 10, 11]). Gutu and Glavan defined the shadowing property for a iterated function system and prove that if a parameterized IFS is uniformly contracting, then it has the shadowing property [11].
Fractals are very important for many sciences like geophysics, geology, image processing,... and iterated function systems is a method for getting fractal sets from dynamical equations [1, 2, 6, 13]. The main motivation of the author is to study various shadowing properties and their relation with stability in iterated function systems [7, 8, 9, 10]. The present paper is indeed one step of this main purpose.
This paper concerns shadowing and weak shadowing properties for IFS's and some important results about weak shadowing property are extended to iterated function systems. Firstly, we introduce the weak shadowing property and conjugacy on **IFS**. Then a sufficient condition for satisfying shadowing property is considered for IFS's. By this result we give two nontrivial systems which has the shadowing property. In Theorem 4.8 we show that if an **IFS** has the shadowing property it has the weak shadowing property. So every uniformly contracting (expanding) **IFS** has the weak shadowing property. In Example 4.5 we give an **IFS**

which has the weak shadowing property but does not have the shadowing property. Then we prove that the weak shadowing is a generic property in $\mathcal{H}_\Lambda(X)$.

## 2. Preliminaries

Let $(X, d)$ be a complete metric space. Let us recall that a *parameterized Iterated Function System(IFS)* $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is any family of continuous mappings $f_\lambda : X \to X,\ \lambda \in \Lambda$, where $\Lambda$ is a finite nonempty set (see [11]).
Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geqslant 0\}$ and $\Lambda^T$ denote the set of all infinite sequences $\{\lambda_i\}_{i\in T}$ of symbols belonging to $\Lambda$. A typical element of $\Lambda^{\mathbb{Z}_+}$ can be denoted as $\sigma = \{\lambda_0, \lambda_1, ...\}$ and we use the shorted notation

$$\mathcal{F}_{\sigma_n} = f_{\lambda_n} o ... o f_{\lambda_1} o f_{\lambda_0}.$$

**Definition 2.1.** [11] *A sequence $\{x_n\}_{n\in T}$ in $X$ is called an orbit of the **IFS** $\mathcal{F}$ if there exists $\sigma \in \Lambda^T$ such that $x_{n+1} = f_{\lambda_n}(x_n)$, for each $\lambda_n \in \sigma$. Given $\delta > 0$, a sequence $\{x_n\}_{n\in T}$ in $X$ is called a $\delta-$pseudo orbit of $\mathcal{F}$ if there exists $\sigma \in \Lambda^T$ such that for every $\lambda_n \in \sigma$, we have $d(x_{n+1}, f_{\lambda_n}(x_n)) < \delta$.*

One says that the **IFS** $\mathcal{F}$ has the *shadowing property* (on $T$) if, given $\epsilon > 0$, there exists $\delta > 0$ such that for any $\delta-$pseudo orbit $\{x_n\}_{n\in T}$ there exists an orbit $\{y_n\}_{n\in T}$, satisfying the inequality $d(x_n, y_n) \leqslant \epsilon$ for all $n \in T$. In this case one says that the $\{y_n\}_{n\in T}$ or the point $y_0$, $\epsilon-$shadows the $\delta-$pseudo orbit $\{x_n\}_{n\in T}$.

**Definition 2.2.** *One says that the **IFS** $\mathcal{F}$ has the* weak shadowing property *(on $T$) if, given $\epsilon > 0$, there exists $\delta > 0$ such that for any $\delta-$pseudo orbit $\boldsymbol{x} = \{x_n\}_{n\in T}$ there exists an orbit $\boldsymbol{y} = \{y_n\}_{n\in T}$, satisfying $\boldsymbol{y} \subset B_\epsilon(\boldsymbol{x})$.*
*Where $B_\epsilon(S)$ denote the set of all $x \in X$ such that $d(x, S) < \epsilon$.*

Please note that if $\Lambda$ is a set with one member then the parameterized **IFS** $\mathcal{F}$ is an ordinary discrete dynamical system. In this case the shadowing property for $\mathcal{F}$ is ordinary shadowing property for a discrete

dynamical system.
The parameterized **IFS** $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is uniformly contracting if there exists

$$\beta = sup_{\lambda \in \Lambda} sup_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x,y)},$$

and this number called also the contracting ratio, is less than one.
Respectively, we shall say that $\mathcal{F}$ is uniformly expanding if

$$\alpha = inf_{\lambda \in \Lambda} inf_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x,y)} > 1.$$

We call $\alpha$ the expanding ratio [11].
Suppose $f, g$ are two homeomorphism on $X$ we define
$d_0(f,g) = \max\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)) : x \in X\}$.
Let $\mathcal{H}_\Lambda(X)$ denote the set of all IFS, $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ such that each $f_\lambda$ is a homeomorphism and for $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$,$\mathcal{G} = \{X; g_\lambda | \lambda \in \Lambda\} \in \mathcal{H}_\Lambda(X)$ Let

$$\rho(\mathcal{F}, \mathcal{G}) = \max_{\lambda, \mu \in \Lambda} \{d(f_\lambda(x), g_\mu(x)), d(f_\lambda^{-1}(x), g_\mu^{-1}(x)) : x \in X\},$$

if $\mathcal{F} \neq \mathcal{G}$ and $\rho(\mathcal{F}, \mathcal{F}) = 0$.
Clearly $\rho$ is a complete metric on $\mathcal{H}_\Lambda(X)$.
We recall that the space $X$ is homogeneous if for $\epsilon > 0$ we can find $\delta > 0$ which if $\{x_1, ..., x_n\}, \{y_1, ..., y_n\} \subset X$ are two sets of disjoint points satisfying $d(x_i, y_i) \leqslant \delta$, for all $1 \leqslant i \leqslant n$, then there exists a homeomorphism $h : X \to X$ with $d_0(h, id_X) \leqslant \epsilon$ and
$h(x_i) = y_i,\ 1 \leqslant i \leqslant n$ [12].

**Definition 2.3.** [9] *Suppose $(X, d)$ and $(Y, d')$ are compact metric spaces and $\Lambda$ is a finite set. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ are two IFS which $f_\lambda : X \to X$ and $g_\lambda : Y \to Y$ are continuous maps for all $\lambda \in \Lambda$. We say that $\mathcal{F}$ is said to be topologically conjugate to $\mathcal{G}$ if there is a homeomorphism $h : X \to Y$ such that $g_\lambda = hof_\lambda oh^{-1}$, for all $\lambda \in \Lambda$. In this case, $h$ is called a topological conjugacy.*

## 3. A Sufficient Condition for Satisfying Shadowing Property

Suppose $X$ is an Euclidean metric space and for every $x \in X$ and $\epsilon > 0$, $B(x,\epsilon) = \{y \in X : d(x,y) \leqslant \epsilon\}$ is the closed $\epsilon-$ball around $x$. Consider the tent maps $f_s : [0,2] \to [0,2]$ where

$$f_s(x) = \begin{cases} sx, & 0 \leqslant x \leqslant \frac{1}{2}, \\ s(2-x), & \frac{1}{2} \leqslant x \leqslant\leqslant 1. \end{cases}$$

In [5] the authors prove that: "Even if $f_s$ does not have the shadowing property, pseudo-orbits of $f_s$ can be shadowed by actual orbits of $f_t$ for some nearby $t > s$."
It is interesting to obtain a similar result for **IFS**'s, so we have the following theorem.

**Theorem 3.1.** *Let $\epsilon,\ \delta > 0$ and $\Lambda$ be a finite set. Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{X; g_\lambda | \lambda \in \Lambda\}$ are two **IFS** such that $B(f_\lambda(x), \epsilon + \delta) \subseteq g_\lambda(B(x,\epsilon))$, for every $\lambda \in \Lambda$ and $x \in X$. Then every $\delta-$pseudo orbit for $\mathcal{F}$ can be $\epsilon-$shadowed by an orbit in $\mathcal{G}$*

**Proof.** Suppose $\{x_i\}_{i \geqslant 0}$ is a $\delta-$pseudo orbit for $\mathcal{F}$, for each $i \geqslant 0$ there exists $\lambda_i \in \Lambda$ such that $d(f_{\lambda_i}(x_i), x_{i+1}) < \delta$. put $W_0 = B(x_0, \epsilon)$ and

$$W_k = W_{k-1} \cap g_{\lambda_0}^{-1} o g_{\lambda_1}^{-1} o ... o g_{\lambda_{k-1}}^{-1}(B(x_k,\epsilon)), \qquad k \geqslant 1.$$

**Claim:** $g_{\lambda_{k-1}} o ... o g_{\lambda_1} o g_{\lambda_0}(W_k) = B(x_k, \epsilon)$, for all $k \geqslant 0$.
So $\{W_k\}_{k \geqslant 0}$ is a decreasing sequence of nonempty compact sets. Then $\bigcap_{k \geqslant 0} W_k \neq \emptyset$. Choose $x \in \bigcap_{k \geqslant 0} W_k$. For each $k \geqslant 0$, $g_{\lambda_{k-1}} o ... o g_{\lambda_1} o g_{\lambda_0}(x) \in B(x_k, \epsilon)$, thus
$d(g_{\lambda_{k-1}} o ... o g_{\lambda_1} o g_{\lambda_0}(x), x_k) < \epsilon$. This implies that $\{x_i\}_{i \geqslant 0}$ is $\epsilon-$shadowed by an orbit in $\mathcal{G}$. □

**Proof of Claim:** We prove by induction on $k$ that for all $k \geqslant 1$, we have

$$g_{\lambda_{k-1}} o ... o g_{\lambda_1} o g_{\lambda_0}(W_k) = B(x_k, \epsilon).$$

Let $k = 1$. This is clear that $g_{\lambda_0}(W_1) \subseteq B(x_1, \epsilon)$. On the other hand if $y \in B(x_1, \epsilon)$ then $d(f_{\lambda_0}(x_0), y) \leqslant d(f_{\lambda_0}(x_0), x_1) + d(x_1, y) < \epsilon + \delta$. So

by hypothesis $y = g_{\lambda_0}(z)$, for some $z \in B(x_0, \epsilon) \cap g_{\lambda_0}^{-1}(B(x_1, \epsilon))$. Indeed $y \in g_{\lambda_0}(W_0 \cap g_{\lambda_0}^{-1}(B(x_1, \epsilon))$. Suppose
$g_{\lambda_{k-1}} o ... o g_{\lambda_1} o g_{\lambda_0}(W_k) = B(x_k, \epsilon)$. This is clear that
$g_{\lambda_k} o ... o g_{\lambda_1} o g_{\lambda_0}(W_{k+1}) \subseteq B(x_{k+1}, \epsilon)$. On the other hand if $y \in B(x_{k+1}, \epsilon)$ then $d(f_{\lambda_k}(x_k), y) \leqslant d(f_{\lambda_k}(x_k), x_{k+1}) + d(x_{k+1}, y) < \epsilon + \delta$.
So $y \in B(f_{\lambda_k}(x_k), \epsilon + \delta) \subseteq g_{\lambda_k}(B(x_k, \epsilon))$. This implies that $y = g_{\lambda_k}(z)$ for some
$z \in B(x_k, \epsilon) \cap g_{\lambda_k}^{-1}(B(x_{k+1}, \epsilon))$. Since $g_{\lambda_{k-1}} o ... o g_{\lambda_1} o g_{\lambda_0}(W_k) = B(x_k, \epsilon)$, then $y \in g_{\lambda_k} o ... o g_{\lambda_1} o g_{\lambda_0}(W_{k+1})$.

**Corollary 3.2.** *The **IFS**, $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ has the shadowing property if, for every $\epsilon > 0$, there is a $\delta > 0$ such that*
*$B(f_\lambda(x), \epsilon + \delta) \subseteq f_\lambda(B(x, \epsilon))$ holds for all $x \in X$ and $\lambda \in \Lambda$.*

Now, we are going to construct examples of iterated function systems and investigate the shadowing property for them.
By using of Corollary 3.2 we show that the following **IFS** on $S^1$ has the shadowing property.

**Example 3.3.** Consider the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. The natural distance on $\mathbb{R}$ induces a distance, $d$, on $S^1$. For $m \in \{2, 3\}$ define the map $f_m : S^1 \to S^1$ by

$$f_m(x) = (mx) \ mod \ 1.$$

By assuming that $\mathcal{G} = \{\mathbb{C}; f_2, f_3\}$ is uniformly expanding, the authors prove that the affine **IFS** $\mathcal{G}$ has the shadowing property; see Theorem 4.1 of [11] for more details. But, since $f_2(\frac{1}{2}) = f_2(0)$ and $f_3(\frac{1}{3}) = f_3(0)$, this is clear that $\mathcal{F} = \{S^1; f_2, f_3\}$ isn't uniformly expanding. So, by using of Corollary 3.2 we show that the following $\mathcal{F}$ has the shadowing property.
For given $\epsilon > 0$ we put $\delta = \frac{\epsilon}{5}$. Let $x, y \in [0, 1)$ and $y \in B(f_2(x), \epsilon + \delta)$. There exist $t, s \in [0, 1)$ that $f_2(t) = y = f_2(s)$(if $y = 0$ then $t = s$, otherwise $t \neq s$). Without less of generality, we assume that
$\mid 2s - 2x \mid \leqslant \epsilon + \delta$. This implies that $\mid s - x \mid \leqslant \frac{\epsilon + \delta}{2} < \epsilon$.

So $y = f_2(s) \in f_2(B(x, \epsilon))$. Similarly $B(f_3(x), \epsilon + \delta) \subseteq f_3(B(x, \epsilon))$. By Corollary 3.2, the **IFS** $\mathcal{F} = \{S^1; f_2, f_3\}$ has the shadowing property.

**Example 3.4.** The sequence space on two symbols is the set

$$\Sigma = \{(x_0, x_1, x_2, \cdots) \mid x_i = 0 \text{ or } 1\}.$$

Let $x = (x_0, x_1, x_2, \cdots)$ and $y = (y_0, y_1, y_2, \cdots)$ be two elements in $\Sigma$. We define the distance between them to be $\rho(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$. This is well known that $(\Sigma, \rho)$ is a compact metric space. Let $\sigma : \Sigma \longrightarrow \Sigma$ be the corresponding shift map, i.e. $(\sigma(x))_i = x_{i+1}$ and $\alpha_2 : \Sigma \to \Sigma$ be a map which is defined by

$$\alpha_2(x_0x_1x_2\cdots) = (x_1x_3x_4x_5\cdots).$$

Take $\mathcal{F} = \{\Sigma; \sigma, \alpha_2\}$. Since $\sigma$ and $\alpha_2$ aren't injective, the **IFS** $\mathcal{F}$ isn't uniforming expanding. We show that $\mathcal{F}$ has the shadowing property.
Let $\epsilon > 0$. There exists an integer number, $N(\epsilon)$, that $\frac{1}{2^{N(\epsilon)}} \leqslant \epsilon < \frac{1}{2^{N(\epsilon)-1}}$. Choose $\delta$ such that $0 < \delta < \frac{1}{2^{N(\epsilon)-1}} - \epsilon$.

Let $x = (x_0x_1x_2x_3\cdots)$ be an arbitrary point in $\Sigma$.
First, we show that $B(\alpha_2(x), \epsilon + \delta) \subseteq \alpha_2(B(x, \epsilon))$. By definition of $\alpha_2$, we have $\alpha_2(x) = (x_1x_3x_4x_5\cdots)$. Suppose $\rho(\alpha_2(x), y) < \epsilon + \delta < \frac{1}{2^{N(\epsilon)-1}}$, so $y_0 = x_1$ and $y_i = x_{i+2}$, for all $0 \leqslant i \leqslant N(\epsilon) - 1$. Now, we consider the point $z = (z_0z_1z_2\cdots)$ such that $z_0 = x_0, z_1 = x_1, z_2 = x_2$ and $z_i = y_{i-2}$, for all $i \geqslant 0$. So $z_i = x_i$, for $i = 0, 1, 2, \cdots, N(\epsilon)$. Then $y = \alpha_2(z) \in \alpha_2(B(x, \epsilon))$. This implies that $B(\alpha_2(x), \epsilon+\delta) \subseteq \alpha_2(B(x, \epsilon))$. Similar argument show that $B(\sigma(x), \epsilon + \delta) \subseteq \sigma(B(x, \epsilon))$. By Corollary 3.2, the **IFS** $\mathcal{F} = \{\Sigma; \sigma, \alpha_2\}$ has the shadowing property.

## 4. Weak Shadowing Property for Iterated Function Systems

In this section, we investigate the structure of parameterized *IFS* with the weak shadowing property. It is well known that if $f : X \to X$ and $g : Y \to Y$ are conjugated then $f$ has the weak shadowing property if and only if so does $g$. In the next theorem we extend this property for *iterated function systems* .

**Theorem 4.1.** *Suppose $(X, d)$ and $(Y, d')$ are compact metric spaces and $\Lambda$ is a finite set. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | i \in \Lambda\}$ are* *two IFS which $f_\lambda : X \to X$ and $g_\lambda : X \to X$ are continuous maps for all $\lambda \in \Lambda$. Suppose that $\mathcal{F}$ is topologically conjugate to $\mathcal{G}$, then $\mathcal{F}$ has the weak shadowing property if and only if so does $\mathcal{G}$.*

**Proof.** Suppose that $\mathcal{F}$ has the weak shadowing property, we prove that $\mathcal{G}$ also have this property. Fix $\epsilon > 0$ and consider $h : X \to Y$ as the conjugacy map between $\mathcal{F}$ and $\mathcal{G}$. Since $h$ is a homeomorphism then there exists $\epsilon_1 > 0$ such that $d(a, b) < \epsilon_1$, implies $d^{'}(h(a), h(b)) < \epsilon$. Let $\delta_1 > 0$ be an $\epsilon_1$ modulus of weak shadowing for $\mathcal{F}$, there is $\delta > 0$ such that $d^{'}(x, y) < \delta$ implies that $d(h^{-1}(x), h^{-1}(y)) < \delta_1$.
Now, Suppose that $\mathbf{x} = \{x_i\}_{i\geqslant 0}$ is a $\delta-$pseudo orbit for $\mathcal{G}$. Then $\mathbf{x}^{'} = \{h^{-1}(x_i)\}_{i\geqslant 0}$ is a $\delta_1-$pseudo orbit for $\mathcal{F}$. Since $\mathcal{F}$ has the weak shadowing property then there exists an orbit $\mathbf{y}^{'} = \{y_i\}_{i\geqslant 0}$ in $\mathcal{F}$ such that $\mathbf{y}^{'} \subset B_{\epsilon_1}(\mathbf{x}^{'})$. So, $\mathbf{y} \subset B_\epsilon(\mathbf{x})$, where $\mathbf{y} = \{h(y_i)\}_{i\geqslant 0}$ is an orbit of $\mathcal{G}$. □

By shadowing and weak shadowing definitions for **IFS** we have the following theorem.

**Theorem 4.2.** *Let $X$ be a complete metric space, if $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ has the shadowing property then it has the weak shadowing property.*

So, Theorem 4.8, Theorems 2.1 and 2.2 in [11] we have the following results.

**Corollary 4.3.** *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is uniformly contracting, then it has the weak shadowing property on $\mathbb{Z}_+$.*

**Corollary 4.4.** *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is uniformly expanding and if each function $f_\lambda(\lambda \in \Lambda)$ is surjective, then the IFS has the weak shadowing property on $\mathbb{Z}_+$.*
The following example shows that the inverse of Theorem is not true.

**Example 4.5.** Consider the unit circle $\mathbb{S}^1$ with coordinate $x \in [0, 1)$. Suppose that $0 < \beta_1, \beta_2 < 1$ are two distinct irrational numbers and $f_i$ is homeomorphisms on $S^1$ defined by $f_i(x) = x + \beta_i$, for $i \in \{0, 1\}$. Let $\mathcal{F} = \{\mathbb{S}^1; f_1, f_2\}$. Since every orbit of $f_1$ is an orbit of $\mathcal{F}$ that is dense in $\mathbb{S}^1$, then $\mathcal{F}$ has the weak shadowing property [17].
Now, suppose that $\beta_2 - \beta_1 = \frac{1}{2}$. We show that $\mathcal{F}$ does not have the

shadowing property.
To obtain a contradiction, we assume that $\mathcal{F}$ has the shadowing property. Take $\epsilon = \frac{1}{5}$ and $\delta > 0$ be the corresponding number for shadowing property. Let $\alpha$ be a rational number which $|\alpha - \beta_1| < \delta$ and
$g : S^1 \to S^1$ be a homeomorphism defined by $g(x) = x + \alpha$. This is clear that every orbits of $g$ is a $\delta-$pseudo orbit of $f_1(x) = x + \beta$. Since $\alpha$ is a rational number, then there is $m \in \mathbb{N}$ such that $g^m$ is identity map. Let $\sigma = \{..., \lambda_{-1}, \lambda_0, \lambda_1, ...\}$ be an arbitrary sequence in $\{1, 2\}^{\mathbb{Z}}$.

**Claim:** *For every $p \in \mathbb{S}^1$, the sets $\{\mathcal{F}_{\sigma_{mk}}(p)\}_{k \geqslant 0}$ is dense in $\mathbb{S}^1$ .*
So, for any $x, p \in \mathbb{S}^1$, $\{x, g(x), ..., g^{m-1}(x), ...\}$ is a $\delta$-pseudo orbit of $\mathcal{F}$, but there exists $k > 0$ such that $d(g^{km}(x), \mathcal{F}_{\sigma_{mk}}(p)) > \frac{1}{5}$.
Hence $\mathcal{F}$ does not have the shadowing property.

**Proof of Claim**. For any $n > 0$, let $A_n = \{\lambda_i; \lambda_i = 2, 1 \leqslant i \leqslant n\}$ and $n_2$ be the cardinality of the set $A_n$. Then
$\mathcal{F}_{\sigma_{mk}}(p) = p + mk\beta_1 + \frac{mk_2}{2} (mod\ 1)$. By a similar argument to that given in ( [14] Example 2), we can show that $\{\mathcal{F}_{\sigma_{mk}}(p)\}_{k \geqslant 0}$ is dense in $\mathbb{S}^1$ .

**Lemma 4.6.** *Suppose $\mathbf{x} = \{x_i\}_{i=0}^n$ is a $\delta-$pseudo orbit. There exists a $2\delta-$pseudo trajectory $\mathbf{y} = \{y_i\}_{i=0}^n$ such that $\mathbf{x} \subset B_\epsilon(\mathbf{y})$ and $y_i \neq y_j$ for all $i \neq j$.*

**Proof.** Consider a finite sequence $\{\lambda_i\}_{i=0}^{n-1} \subset \Lambda$ such that
$d(f_{\lambda_i}(x_i, x_{i+1}) < \delta$, for all $0 \leqslant i \leqslant n - 1$. Since $X$ is a compact space and has no isolated points then we can find a sequence $\mathbf{y} = \{y_i\}_{i=0}^n$ of distinct points such that $d(f_\lambda(x_i), y_i) < \frac{\delta}{2}$ and $d(x_{i+1}, y_{i+1}) < \frac{\delta}{2}$, for all $\lambda \in \Lambda$ and all $0 \leqslant i \leqslant n - 1$. So
$d(f_{\lambda_i}(y_i), y_{i+1}) \leqslant d(f_{\lambda_i}(y_i), f_{\lambda_i}(x_i)) + d(f_{\lambda_i}(x_i), x_{i+1}) + d(x_{i+1}, y_{i+1}) < \frac{\delta}{2} + \frac{\delta}{2} + \delta$, for $0 \leqslant i \leqslant n - 1$. □

**Lemma 4.7.** *Let $\mathrm{x} = \{x_i\}_{i \in \mathbb{Z}}$ be a sequence in $X$ and $\mathrm{x}_n = \{x_i\}_{i=-n}^n$, for every $n \geqslant 1$. For each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $\mathrm{x} \subset B_\epsilon(\mathrm{x}_k)$.*

**Proof.** This is clear that $\{X - \overline{\mathrm{x}}, B_\epsilon(\mathrm{x}_n) : \ n \geqslant 1\}$ where $\overline{\mathrm{x}}$ is closure of x, is an open cover for $X$ and has a finite subcover. Suppose
$X \subset (X - \overline{\mathrm{x}}) \cup B_\epsilon(\mathrm{x}_{n_1}) \cup ... \cup B_\epsilon(\mathrm{x}_{n_l})$, for some $n_1 < n_2 < ... < n_l$. Since $\mathrm{x} \cap (X - \overline{\mathrm{x}}) = \emptyset$ and

$B_\epsilon(\mathrm{x}_{n_1}) \subset B_\epsilon(\mathrm{x}_{n_2}) \subset ... \subset B_\epsilon(\mathrm{x}_{n_l})$, then $\mathrm{x} \subset B_\epsilon(\mathrm{x}_{n_l})$. $\square$

Next theorem is the main result of this paper and the main idea of proof is the same as that of [12].

**Theorem 4.8.** *If the space $X$ is a generalized homogeneous then the weak shadowing property is generic in $\mathcal{H}_\Lambda(X)$.*

**Proof.** Given $\epsilon > 0$ and $V = \{V_1, ..., V_k\}$ be a cover of $X$ consisting of open sets with diameters less than $\epsilon$. Suppose $\mathcal{F} \in \mathcal{H}_\Lambda(X)$ and take $J_\mathcal{F}$ is the family of sets $L \subset \{1, 2, ..., k\}$ such that there exists an obit $\mathrm{x} = \{x_i\}_{i\in\mathbb{Z}}$ of $\mathcal{F}$ satisfying $\mathrm{x} \cap V_j \neq \emptyset$, for all $j \in L$.

**Claim.** For any $\mathcal{F} \in \mathcal{H}_\Lambda(X)$ there is a neighborhood $U$ of $\mathcal{F}$ such that $J_\mathcal{F} \subset J_\mathcal{G}$ for $\mathcal{F} \in U$. Take

$$C_V = \{\mathcal{F} \in \mathcal{H}_\Lambda(X) : J_\mathcal{F} = J_\mathcal{G} for\ \mathcal{G}\ sufficiently\ close\ to\ \mathcal{F}\}.$$

By definition $C_V$ is an open subset of $\mathcal{H}_\Lambda(X)$. Now we show that $C_V$ is dense in $\mathcal{H}_\Lambda(X)$. Consider an arbitrary open set $W \subset \mathcal{H}_\Lambda(X)$ and $J_\mathcal{F}$ is a maximal element of $J_W = \{J_\mathcal{F} : \mathcal{F} \in W\}$, i.e., for every $\mathcal{G} \in W$, $J_\mathcal{F} \subset J_\mathcal{G}$ implies that $\mathcal{F} = \mathcal{G}$. Thus, by claim $\mathcal{F} \in C_V \cap W$. So $C_V$ is an open dense subset of $\mathcal{H}_\Lambda(X)$.
Take $\mathcal{F} \in C_V$ we prove that $\mathcal{F}$ has the weak shadowing property.
Since $C_V$ is an open set there is $\gamma > 0$ such that for $\mathcal{G} \in \mathcal{H}_\Lambda(X)$, $\rho(\mathcal{F}, \mathcal{G}) < \gamma$ implies that $J_\mathcal{F} = J_\mathcal{G}$. Suppose $\beta > 0$ is a $\gamma-$modulus of homogeneity of $X$. Let $\mathrm{x} = \{x_i\}_{i\in\mathbb{Z}}$ be a $\frac{\beta}{2}-$pseudo orbit of $\mathcal{F}$. Because of the Lemma 4.7 there exists $k \in \mathbb{N}$ such that $\mathrm{x} \subset B_\epsilon(\mathrm{x}_k)$. By Lemma 4.6 there exists a $\beta-$pseudo trajectory $\mathbf{y} = \{y_i\}_{i=-n}^{n}$ ( belong to $\mathcal{F}$) such that $\mathbf{x_k} \subset B_\epsilon(\mathbf{y})$ and $y_i \neq y_j$ for all $i \neq j$. Suppose $0 < \tau < \gamma$ is a number that $d(a, b) < \tau$ implies that $d(f_\lambda^{-1}(a), f_\lambda^{-1}(b)) < \gamma$, for all $a, b \in X$ and all $\lambda \in \Lambda$. Also, suppose $h \in \mathcal{H}(X)$, $d_0(h, id_X) < \tau$ is a homeomorphism connecting $f_{\lambda_i}(y_i)$ with $y_{i+1}$ for all $-n \leqslant i \leqslant n-1$. Consider IFS $\mathcal{G} = \{X; g_\lambda = hof_\lambda | \lambda \in \Lambda\}$ and let $\sigma = \{\mu_0, \mu_1, ...\}$ be an arbitrary element of $\Lambda^{\mathbb{Z}_+}$. So the sequence

$$\mathbf{z} = \{..., g_{\mu_1}^{-1}(g_{\mu_0}^{-1}(y_{-n})), g_{\mu_0}^{-1}(y_{-n}), y_{-n}, y_{-(n-1)}, ...,$$
$$y_n, g_{\mu_0}(y_n), g_{\mu_1}(g_{\mu_0}(y_n)), ...\},$$

is an orbit of $\mathcal{G}$.
This is clear that $\rho(\mathcal{F},\mathcal{G}) < \gamma$ and hence $J_{\mathcal{F}} = J_{\mathcal{G}}$. So there ia an orbit $\mathbf{z}^{'}$ of $\mathcal{F}$ such that for any $1 \leqslant i \leqslant k$ if $\mathbf{z}^{'} \cap V_i \neq \emptyset$ then $\mathbf{z} \cap V_i \neq \emptyset$. Thus $\mathbf{z} \subset B_\epsilon(\mathbf{z}^{'})$ and consequently $\mathbf{z}^{'} \subset B_{3\epsilon}(\mathbf{x})$. □

**Proof of Claim**: Suppose that $J_F = \{L_1, L_2, ..., L_m\}$. For any $1 \leqslant j \leqslant m$, there is an orbit $\mathrm{x}^j$ such that $\mathrm{x}^j \cap U_{j_i} \neq \emptyset$ for all $j_i \in L_j$. So there exists $\epsilon_j > 0$ such that $\rho(\mathcal{F},\mathcal{G}) < \epsilon_j$ implies that, for an orbit $\mathrm{y}^j$ of $\mathcal{G}$ such that $\mathrm{y}^j \cap U_{j_i} \neq \emptyset$ for all $j_i \in L_j$. Thus $L_j \in$. Take $\epsilon = \min\{\epsilon_1, \epsilon_2, ..., \epsilon_m\}$, similar argument shows that if $\rho(\mathcal{F},\mathcal{G}) < \epsilon$ then $L_1 \in J_G$ for all $1 \leqslant j \leqslant m$.
By Theorem 4.8 and proof of Theorem 2 in [12] we have the following theorem.

**Theorem 4.9.** *If the space $X$ is one of the following:*
*(i) a topological manifold with boundary (dim$(X) \geqslant 2$ if $\partial X \neq \emptyset$.),*
*(ii) a Cartesian product of a countably infinite number of manifolds with nonempty boundary,*
*(iii) a Cantor set,*
*then weak shadowing for **IFS** is a generic property in $H_\Lambda(X)$.*

## 5. Conclusion

In this paper, we give a sufficient condition for an **IFS** to have the shadowing property. By using of this result, Example 3.3 gives a non-trivial **IFS** on the unit circle $S^1$, which has the shadowing property. Then, weak shadowing property for **IFS**'s is considered and this proved that, for this systems, shadowing implies weak shadowing but the converse is not necessarily true. The topological conjugacy is considered for **IFS**, the importance of this concept is indicated by the fact that: If $\mathcal{F}$ and $\mathcal{G}$ are two conjugated **IFS** then $\mathcal{F}$ has the weak shadowing property if and only if so does $\mathcal{G}$.
Section 4. is devoted to define generic property for **IFS**'s and the problem of genericity of the weak shadowing property when the dimension of the manifold is arbitrary. Theorem 4.9 provide some conditions to the weak shadowing be a generic property. Investigating the relation

between chaos and stability with weak shadowing for iterated function systems will become our future research topic.

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