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Shifted Chebyshev polynomial method for solving systems of linear and nonlinear Fredholm-Volterra integro-differential equations

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Abstract. This paper suggests a novel and efficient method for solving systems of Fredholm-Volterra integro-differential equations (FVIDEs). A Chebyshev matrix approach is implemented for solving linear and nonlinear FVIDEs under initial boundary conditions. The aim of this work is to construct a quick and precise numerical approximation by a simple, tasteful and powerful algorithm based on the Chebyshev series representation for solving such systems. The properties of shifted Chebyshev polynomials are used to transform the system of FVIDEs into a system of algebraic equations. Then, the corresponding matrix equation will be solved by using the Galerkin-like procedure to find the unknown coefficients which are related to the approximate solution. Also, the polynomial convergence rate of our method is discussed by preparing some theorems and lemmas. Finally, some numerical examples are given to illuminate the reliability and high accuracy of this algorithm in comparison with some other well-known methods.

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1 Introduction

Systems of linear and nonlinear integro-differential equations and their solutions play a major role in science and engineering. A physical event can be modeled as a differential equation, an integral equation or an integro-differential equation or a system of these equations. Since most of these equations cannot be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation.

Fredholm integro-differential equations (FIDEs) and FVIDEs are encountered in model problems of science and engineering. Such kind of equations arises in the mathematical modeling of various physical phenomena, such as the heat conduction, the materials with memory and the combined conduction, convection and radiation problems. In the past decades, both mathematicians and physicists have devoted considerable effort to find robust and stable numerical methods for solving systems of linear and nonlinear FIDEs and FVIDEs with physical interest. Numerical and analytical methods have included Haar functions method [17], the Legendre matrix method [21], the Lagrange method [25], the differential transformation method [5], the Tau method [19], the Chebyshev polynomial method [4], the Bernstein operational matrix approach [15], the Taylor expansion method [13], and the operational Tau method [1]. However, several numerical methods have been used to approximate the solution of FVIDEs such as the collocation method [10], Legendre collocation method [18], Taylor polynomial method [22], Least squares method [23] and so on.

Since 1994, the Taylor, Chebyshev, Legendre, Bessel, Hermite, Laguerre, Bernoulli and Bernstein matrix methods have been used to solve high-order linear and nonlinear (including hyperbolic partial differential equations) Fredholm-Volterra integro-differential equations [6, 3, 28, 8, 14, 27].

Up to now, several numerical methods have been used for solving FVIDEs with different forms of bases. In most of these articles, to approximate the solution of these equations, the bases of Legendre, Bernstein, $\cdots$ are used. Recently, some authors applied these methods for solving systems of FIDEs, where most of them consists two types of Volterra and Fredholm- Volterra equations. In this paper, we propose an efficient method to approximate the solution of system of FVIDEs. To do this, at first the N-th truncation of Chebyshev series for unknown functions y_j is substituted instead of the unknown function in the given FVIDEs. Then, a system of linear equations with unknown Chebyshev coefficients will be obtained. In the next step, these unknown coefficients will be determined by using the Galerkin-like procedure [9]. A violent mathematical proof is provided for the error analysis and convergence of this approach.

In this work, the shifted Chebyshev Galerkin matrix method is presented for solving the system of high-order linear FVIDEs in the form

$$\sum_{n=0}^{n_1}\sum_{j=1}^{k} P_{i,j,n}(x)y_j^{(n)}(x) = g_i(x) + \int_a^b \sum_{n=0}^{n_2}\sum_{j=1}^{k} \kappa^f_{i,j,n}(x,\gamma)y_j^{(n)}(\gamma)d\gamma$$
$$+\int_a^x \sum_{n=0}^{n_3}\sum_{j=1}^{k} \kappa^v_{i,j,n}(x,\gamma)y_j^{(n)}(\gamma)d\gamma, a \le x,\gamma \le b,\ i=1,...,k,\ n_1 \ge n_2, n_3, \tag{1}$$

with the initial-boundary conditions

$$\sum_{j=0}^{n_1-1}\left[\alpha_{i,j,n}y_n^{(j)}(a) + \beta_{i,j,n}y_n^{(j)}(b)\right] = \mu_{n,i},\ i=0,1,...,n_1-1,\ n=1,2,...,k, \tag{2}$$

where $P_{i,j,n}(x)$ and $g_i(x)$ are given continuous functions in $L^2[(a,b)]$, $\kappa^f_{i,j,n}(x,\gamma)$ (f is the abbreviation of Fredholm part) and $\kappa^v_{i,j,n}(x,\gamma)$ (v is the abbreviation of Volterra part) are given sufficiently smooth kernel functions in $L^2[(a,b)\times(a,b)]$. Also, $y_n^{(j)}(a)$, $y_n^{(j)}(b)$ and $\mu_{n,i}$ are real constants and $y_j^{(0)} = y_j$, $j=1,2,...,k$ are unknown functions that will be determined. Moreover, we extend our scheme for solving the following nonlinear FVIDEs

$$\sum_{n=0}^{n_1}\sum_{j=1}^{k} P_{i,j,n}(x)y_j^{(n)}(x) = g_i(x) + \int_a^b \sum_{n=2}^{n_2}\sum_{j=1}^{k} \kappa^f_{i,j,n}(x,\gamma)(y_j(\gamma))^n d\gamma$$
$$+\int_a^x \sum_{n=2}^{n_3}\sum_{j=1}^{k} \kappa^v_{i,j,n}(x,\gamma)(y_j(\gamma))^n d\gamma, a \le x,\gamma \le b,\ i=1,...,k, \tag{3}$$

with the same initial-boundary conditions.

Our aim is to find an approximate solution of Eq. (1) under the mixed conditions (2) which can be expressed in the following truncated Chebyshev series form

$$^{N}y_i(x) = \sum_{n=0}^{N} a_{i,n} T_n^*(x),\ i = 1, 2, ..., k,\ a \leq x \leq b, \tag{4}$$

where N will be chosen such that the approximate solution (4) fulfills the initial-boundary conditions (2), i.e. $N \geq n_1$. Also, $T_n^*(x)$ are shifted Chebyshev polynomials on $[a, b]$, that satisfy the following formula

$$T_n^*(x) = T_n\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right),\ a \leq x \leq b,\ n \geq 0,$$

where

$$T_0(x) = 1,\ T_1(x) = x,\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n \geq 2,\ -1 \leq x \leq 1.$$

2 The shifted Chebyshev Galerkin method for FVIDEs

In this section, we will construct the matrix form of each term of equation (1). Assume that Eq. (1) under conditions (2) have a unique solution. We can write the approximate solution $y_j(x)$ in the following matrix form

$$\left[^{N}y_j(x)\right] = \mathbf{T}^*(x)\mathbf{A}_j, \tag{5}$$

where

$$\mathbf{T}^*(x) = \begin{bmatrix} T_0^*(x) & \cdots & T_N^*(x) \end{bmatrix},\ \mathbf{A}_j = \begin{bmatrix} a_{j,0} & \cdots & a_{j,N} \end{bmatrix}^T,\ j = 1, 2, ...k.$$

The n-th order derivative of this solution is

$$\left[^{N}y_j^{(n)}(x)\right] = \mathbf{T}^{*(n)}(x)\mathbf{A}_j. \tag{6}$$

Let $\mathbf{X}(x) = \begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix}$. Then, one can write the matrix form of $\mathbf{T}^*(x)$ as follows

$$\mathbf{T}^*(x) = \mathbf{X}(x)\mathbf{D}^T, \tag{7}$$

where $\mathbf{D}$ is the $(N+1)\times(N+1)$ matrix coefficients defined by

$$\mathbf{D}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{a+b}{(a-b)^1} & -\frac{2}{a-b} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \sum\limits_{k=0}^{N}\frac{a^{N-k}b^k\binom{2N}{2k}}{(a-b)^N} & -N\sum\limits_{k=0}^{N-1}\frac{a^{N-(k+1)}b^k\binom{2N}{2k+1}}{(a-b)^N} & \cdots & (-1)^N\frac{2^{2N-1}}{(a-b)^N} \end{bmatrix}.$$

The n-th order derivative of expression (7) is

$$\mathbf{T}^{*^{(n)}}(x)=\mathbf{X}^{(n)}(x)\mathbf{D}^T. \tag{8}$$

The relationship between matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(n)}(x)$ is

$$\mathbf{X}^{(n)}(x)=\mathbf{X}(x)\left(\mathbf{B}^T\right)^n, \tag{9}$$

where

$$\mathbf{B}^T=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(N+1)(N+1)},$$

and $\left(\mathbf{B}^T\right)^0=[\mathbf{I}]_{(N+1)(N+1)}$ is the unit matrix. Substituting (9) in relation (8), we get the matrix representation of $\mathbf{T}^{*^{(n)}}(x)$ in the following form

$$\mathbf{T}^{*^{(n)}}(x)=\mathbf{X}(x)\left(\mathbf{B}^T\right)^n\mathbf{D}^T. \tag{10}$$

Replacing relations (7) and (10) in Eqs. (5) and (6), respectively, one obtain

$${}^N y_j(x)=\mathbf{X}(x)\mathbf{D}^T\mathbf{A}_j,\;\; j=1,2,...,k,$$

and

$${}^N y_j^{(n)}(x)=\mathbf{X}(x)\left(\mathbf{B}^T\right)^n\mathbf{D}^T\mathbf{A}_j,\;\; j=1,2,...,k,\;\; n=0,1,...,n_1.$$

Therefore, the matrices ${}^{N}\mathbf{y}^{(n)}(x),\ n = 0, 1, ..., n_1$ can be expressed as follows

$$ {}^{N}\mathbf{y}^{(n)}(x) = \mathbf{X}(x)\big(\bar{\mathbf{B}}\big)^{n}\bar{\mathbf{D}}\mathbf{A}, \tag{11}$$

where

$$ {}^{N}\mathbf{y}^{(n)}(x) = \begin{bmatrix} {}^{N}y_1^{(n)}(x) \\ {}^{N}y_2^{(n)}(x) \\ \vdots \\ {}^{N}y_k^{(n)}(x) \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix},\ \bar{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T & 0 & \cdots & 0 \\ 0 & \mathbf{D}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^T \end{bmatrix}_{k\times k}, $$

$$ \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 & \cdots & 0 \\ 0 & \mathbf{B}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}^T \end{bmatrix}_{k\times k},\ \bar{\mathbf{X}}(x) = \begin{bmatrix} \mathbf{X}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x) \end{bmatrix}_{k\times k}. $$

System (1) can be written in the following matrix form

$$ \sum_{n=0}^{n_1} \mathbf{P}_n(x)^{N}\mathbf{y}^{(n)}(x) = \mathbf{g}(x) + \mathbf{F}(x) + \mathbf{V}(x), \tag{12}$$

where

$$ \mathbf{P}_n(x) = \begin{bmatrix} p_{1,1,n}(x) & p_{1,2,n}(x) & \cdots & p_{1,k,n}(x) \\ p_{2,1,n}(x) & p_{2,2,n}(x) & \cdots & p_{2,k,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{k,1,n}(x) & p_{k,2,n}(x) & \cdots & p_{k,k,n}(x) \end{bmatrix},\ {}^{N}\mathbf{y}^{(n)}(x) = \begin{bmatrix} {}^{N}y_1^{(n)}(x) \\ {}^{N}y_2^{(n)}(x) \\ \vdots \\ {}^{N}y_k^{(n)}(x) \end{bmatrix},\ \mathbf{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}, $$

$$ \mathbf{F}(x) = \sum_{n=0}^{n_2}\int_a^b \boldsymbol{\kappa}_n^f(x,\gamma)^{N}\mathbf{y}^{(n)}(\gamma)d\gamma,\ \mathbf{V}(x) = \sum_{n=0}^{n_3}\int_a^x \boldsymbol{\kappa}_n^v(x,\gamma)^{N}\mathbf{y}^{(n)}(\gamma)d\gamma, $$

$$ \boldsymbol{\kappa}_n^f(x,\gamma) = \begin{bmatrix} \kappa^f_{1,1,n} & \kappa^f_{1,2,n} & \cdots & \kappa^f_{1,k,n} \\ \kappa^f_{2,1,n} & \kappa^f_{2,2,n} & \cdots & \kappa^f_{2,k,n} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa^f_{k,1,n} & \kappa^f_{k,2,n} & \cdots & \kappa^f_{k,k,n} \end{bmatrix},\ \boldsymbol{\kappa}_n^v(x,\gamma) = \begin{bmatrix} \kappa^v_{1,1,n} & \kappa^v_{1,2,n} & \cdots & \kappa^v_{1,k,n} \\ \kappa^v_{2,1,n} & \kappa^v_{2,2,n} & \cdots & \kappa^v_{2,k,n} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa^v_{k,1,n} & \kappa^v_{k,2,n} & \cdots & \kappa^v_{k,k,n} \end{bmatrix}, \tag{13}$$

$$ \mathbf{F}(x) = \begin{bmatrix} I_1(x) & I_2(x) & \cdots & I_k(x) \end{bmatrix}^T,\ \mathbf{V}(x) = \begin{bmatrix} V_1(x) & V_2(x) & \cdots & V_k(x) \end{bmatrix}^T, $$

and

$$I_i(x) = \int_a^b \sum_{n=0}^{n_2} \sum_{j=1}^{k} \kappa_{i,j,n}^f(x,\gamma) y_j^{(n)}(\gamma) d\gamma,\ V_i(x) = \int_a^x \sum_{n=0}^{n_3} \sum_{j=1}^{k} \kappa_{i,j,n}^v(x,\gamma) y_j^{(n)}(\gamma) d\gamma.$$

The function $P_{i,j,n}(x)$, can be approximated by Chebyshev matrix form as follows

$$P_{i,j,n}(x) = \sum_{l=0}^{N} p_{i,j,n,l} T_l^*(x), \tag{14}$$

where

$$p_{i,j,n,0} = \frac{2}{\pi(b-a)} \int_a^b \left(1 - \left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2\right)^{-\frac{1}{2}} P_{i,j,n}(x) dx,$$

and

$$p_{i,j,n,l} = \frac{4}{\pi(b-a)} \int_a^b \left(1 - \left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2\right)^{-\frac{1}{2}} P_{i,j,n}(x) T_l^*(x) dx,\ l = 1, 2, ..., N.$$

Also, Eq. (14) can be written as

$$P_{i,j,n}(x) = \mathbf{T}^*(x) \tilde{\mathbf{P}}_{i,j,n}, \tag{15}$$

where

$$\tilde{\mathbf{P}}_{i,j,n} = \left[\begin{array}{cccc} p_{i,j,n,0} & p_{i,j,n,1} & \cdots & p_{i,j,n,N} \end{array}\right]^T.$$

Using relation (7), Eq. (15) becomes

$$p_{i,j,n}(x) = \mathbf{X}(x) \mathbf{D}^T \tilde{\mathbf{P}}_{i,j,n}.$$

Therefore,

$$\mathbf{P}_n(x) = \left[\begin{array}{cccc} \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{1,1,n} & \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{1,2,n} & \cdots & \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{1,k,n} \\ \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{2,1,n} & \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{2,2,n} & \cdots & \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{2,k,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{k,1,n} & \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{k,2,n} & \cdots & \mathbf{X}(x)\mathbf{D}^T\tilde{\mathbf{P}}_{k,k,n} \end{array}\right].$$

For $j = 1, ..., k$, the function $g_j(x)$ can be approximated by the Chebyshev matrix form as follows

$$g_j(x) = \sum_{l=0}^{N} g_{j,l} T_l^*(x), \tag{16}$$

where

$$g_{j,0} = \frac{2}{\pi(b-a)} \int_a^b \left(1 - \left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2\right)^{-\frac{1}{2}} g_j(x)dx,$$

and

$$g_{j,l} = \frac{4}{\pi(b-a)} \int_a^b \left(1 - \left(\frac{2x}{b-a} - \frac{b+a}{b-a}\right)^2\right)^{-\frac{1}{2}} g_j(x)T_l^*(x)dx,\ l = 1, 2, ..., N.$$

Eq. (16), can be written as

$$g_j(x) = \mathbf{T}^*(x)\mathbf{G}_j,\ \ j = 1, 2, ..., k, \tag{17}$$

where

$$\mathbf{G}_j = \begin{bmatrix} g_{j,0} & g_{j,1} & \cdots & g_{j,N} \end{bmatrix}^T.$$

Using relation (7), Eq. (17) becomes

$$g_j(x) = \mathbf{X}(x)\mathbf{D}^T\mathbf{G}_j,$$

so

$$\mathbf{g}(x) = \begin{bmatrix} \mathbf{X}(x)\mathbf{D}^T\mathbf{G}_1 & \mathbf{X}(x)\mathbf{D}^T\mathbf{G}_2 & \cdots & \mathbf{X}(x)\mathbf{D}^T\mathbf{G}_k \end{bmatrix}^T.$$

Also, the kernel functions can be approximated by the truncated Chebyshev series

$$\kappa^f_{i,j,n}(x,t) = \sum_{m=0}^{N}\sum_{r=0}^{N} \left({}^C\kappa^f_{i,j,n,m,r}\right) T^*_m(x)T^*_r(t) = \mathbf{T}^*(x)\,{}^C\boldsymbol{\kappa}^f_{i,j,n}\mathbf{T}^{*T}(t), \tag{18}$$

and

$$\kappa^v_{i,j,n}(x,t) = \sum_{m=0}^{N}\sum_{r=0}^{N} \left({}^C\kappa^v_{i,j,n,m,r}\right) T^*_m(x)T^*_r(t) = \mathbf{T}^*(x)\,{}^C\boldsymbol{\kappa}^v_{i,j,n}\mathbf{T}^{*T}(t), \tag{19}$$

where

$${}^C\boldsymbol{\kappa}^f_{i,j,n} = \left[{}^C\kappa^f_{i,j,n,m,r}\right],\ {}^C\boldsymbol{\kappa}^v_{i,j,n} = \left[{}^C\kappa^v_{i,j,n,m,r}\right],\ m, r = 0, ..., N,\ i, j = 1, ..., k.$$

Considering relations (18), (19) and (7), one obtains the matrix form of $\kappa^f_{i,j,n}(x,t)$ and $\kappa^v_{i,j,n}(x,t)$ as follows

$$\kappa^f_{i,j,n}(x,t) = \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^f_{i,j,n}\mathbf{D}\mathbf{X}^T(t), \tag{20}$$

$$\kappa^v_{i,j,n}(x,t) = \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^v_{i,j,n}\mathbf{D}\mathbf{X}^T(t). \tag{21}$$

Therefore, we have

$$\boldsymbol{\kappa}^f_n(x,t) = \begin{bmatrix} \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^f_{1,1,n}\mathbf{D}\mathbf{X}^T(t) & \cdots & \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^f_{1,k,n}\mathbf{D}\mathbf{X}^T(t) \\ \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^f_{2,1,n}\mathbf{D}\mathbf{X}^T(t) & \cdots & \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^f_{2,k,n}\mathbf{D}\mathbf{X}^T(t) \\ \vdots & \ddots & \vdots \\ \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^f_{k,1,n}\mathbf{D}\mathbf{X}^T(t) & \cdots & \mathbf{X}(x)\mathbf{D}^{TC}\boldsymbol{\kappa}^f_{k,k,n}\mathbf{D}\mathbf{X}^T(t) \end{bmatrix},$$

and matrix $\boldsymbol{\kappa}^v_n(x,t)$ is defined in a similar way.

Considering Eqs. (11), (12), (13), (20) and (21), one obtains

$$\begin{bmatrix} \sum_{n=0}^{n_1}\mathbf{P}_n(x)\mathbf{X}(x)\big(\bar{\mathbf{B}}\big)^n\bar{\mathbf{D}} - \sum_{n=0}^{n_2}\int_a^b \boldsymbol{\kappa}^f_n(x,\gamma)\mathbf{X}(x)\big(\bar{\mathbf{B}}\big)^n\bar{\mathbf{D}}d\gamma \\ -\sum_{n=0}^{n_3}\int_a^x \boldsymbol{\kappa}^v_n(x,\gamma)\mathbf{X}(x)\big(\bar{\mathbf{B}}\big)^n\bar{\mathbf{D}}d\gamma \end{bmatrix}\mathbf{A} = \mathbf{g}(x).$$

We suppose that

$$\mathbf{M} = \sum_{n=0}^{n_1}\mathbf{P}_n(x)\mathbf{X}(x)\big(\bar{\mathbf{B}}\big)^n\bar{\mathbf{D}} - \sum_{n=0}^{n_2}\int_a^b \boldsymbol{\kappa}^f_n(x,\gamma)\mathbf{X}(x)\big(\bar{\mathbf{B}}\big)^n\bar{\mathbf{D}}d\gamma$$
$$-\sum_{n=0}^{n_3}\int_a^x \boldsymbol{\kappa}^v_n(x,\gamma)\mathbf{X}(x)\big(\bar{\mathbf{B}}\big)^n\bar{\mathbf{D}}d\gamma, \tag{22}$$

which is in the components form as follows

$$\mathbf{M} = \big[m_1(x) \quad m_2(x) \quad \cdots \quad m_k(x)\big]^T.$$

Taking the inner product of Eq. (22) with $T^*_j(x)$, $0 \le j \le N$, one can find the following relations

$$\big\langle \mathbf{M}, T^*_j(x)\big\rangle = \big\langle \mathbf{g}, T^*_j(x)\big\rangle \Leftrightarrow \begin{bmatrix} \int_a^b m_1(x)T^*_j(x)dx \\ \vdots \\ \int_a^b m_k(x)T^*_j(x)dx \end{bmatrix} = \begin{bmatrix} \int_a^b g_1(x)T^*_j(x)dx \\ \vdots \\ \int_a^b g_k(x)T^*_j(x)dx \end{bmatrix}, \tag{23}$$

to construct a linear system with $k(N+1)$ algebraic equations and $k(N+1)$ unknowns Chebyshev coefficients. Hence, the fundamental matrix Eq. (23) corresponding to Eq. (1) can be written in the form of

$$\mathbf{WA} = \mathbf{G}, \tag{24}$$

where

$$\mathbf{W} = \left[\int_a^b m_1(x)T_j^*(x)dx \quad \int_a^b m_2(x)T_j^*(x)dx \quad \cdots \quad \int_a^b m_k(x)T_j^*(x)dx\right]^T,$$

and

$$\mathbf{G} = \left[\int_a^b g_1(x)T_j^*(x)dx \quad \int_a^b g_2(x)T_j^*(x)dx \quad \cdots \quad \int_a^b g_k(x)T_j^*(x)dx\right]^T.$$

The initial-boundary conditions (2), yields

$$\sum_{j=0}^{n_1-1} \left[\alpha_j \bar{\mathbf{X}}(a) + \beta_j \bar{\mathbf{X}}(b)\right] \left(\bar{\mathbf{B}}^T\right)^j \bar{\mathbf{D}}\mathbf{A} = \mu,$$

where

$$\alpha_j = \begin{bmatrix} \alpha_j^1 & 0 & \cdots & 0 \\ 0 & \alpha_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_j^k \end{bmatrix},\ \beta_j = \begin{bmatrix} \beta_j^1 & 0 & \cdots & 0 \\ 0 & \beta_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_j^k \end{bmatrix},\ \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix},$$

$$\mu_i = \begin{bmatrix} \mu_{i,0} \\ \mu_{i,1} \\ \vdots \\ \mu_{i,n_1-1} \end{bmatrix},\quad \alpha_j^n = \begin{bmatrix} \alpha_{0,j,n} \\ \alpha_{1,j,n} \\ \vdots \\ \alpha_{n_1-1,j,n} \end{bmatrix},\quad \beta_j^n = \begin{bmatrix} \beta_{0,j,n} \\ \beta_{1,j,n} \\ \vdots \\ \beta_{n_1-1,j,n} \end{bmatrix},$$

for $n = 1, 2, ..., k$. So, these initial-boundary conditions can be written in the following matrix form

$$\mathbf{UA} = \mu \text{ or } \left[\mathbf{U}; \mu\right], \tag{25}$$

where

$$\mathbf{U} = \sum_{j=0}^{n_1-1} \left[\alpha_j \bar{\mathbf{X}}(a) + \beta_j \bar{\mathbf{X}}(b)\right] \left(\bar{\mathbf{B}}^T\right)^j \bar{\mathbf{D}}.$$

Finally, to obtain the solution of Eq. (1) under the conditions (2) by replacing the rows of matrix $\mathbf{U}$ and μ by the rows of the matrices $\mathbf{W}$ and $\mathbf{G}$, respectively, we get

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}}. \tag{26}$$

For convenience, if the last $n_1 k$ rows of the matrix $\mathbf{W}$ are replaced, we have the new augmented matrix form as follows

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right] = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,k(N+1)} & g_{1,0}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k(N-n_1+1),1} & w_{k(N-n_1+1),2} & \cdots & w_{k(N-n_1+1),k(N+1)} & g_{k,k(N-n_1+1)}(x) \\ u_{1,1} & u_{1,2} & \cdots & u_{1,k(N+1)} & \mu_{1,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n_1k,1} & u_{n_1k,2} & \cdots & u_{n_1k,k(N+1)} & \mu_{k,n_1-1} \end{bmatrix}, \tag{27}$$

where, $g_{i,j} = \int_a^b g_i(x)T_j^*(x)dx$. However, we do not have to replace the last rows. For example, if the matrix $\mathbf{W}$ is singular, then the rows that are linear dependent or all zeros are replaced. If $rank\tilde{\mathbf{W}} = rank[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = k(N+1)$, then matrix $\mathbf{A}$ (thereby the unknown Chebyshev coefficients) is uniquely determined. Therefore, system (1) with the initial-boundary conditions (2) has a unique solution. It should be noted that, if matrix $\tilde{\mathbf{W}}$ is singular and $rank\tilde{\mathbf{W}} = rank[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] < k(N+1)$, then we may find a specific solution. Otherwise if $rank\tilde{\mathbf{W}} \neq rank[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] < k(N+1)$, then there is no solution.

By solving algebraic system (27), we can find the Chebyshev coefficients $a_{i,n}$, $i = 1,...,k$, $n = 0,...,N$. Eventually, substituting these Chebyshev coefficients in (4), the required approximate solution for the FVIDEs (1) will be determined.

Remark 2.1. Note that for solving (3) we apply a similar procedure, and because of exhibiting additional discussions we remove explaining the method of solution in this case. Moreover, the associated algebraic system of (3) is a nonlinear one instead of linear.

3 Convergence analysis

The goal of this section is to analyze the proposed approximation scheme for numerical solution of FVIDEs (1). In order to study the convergence analysis of this method we will give some useful lemmas, theorems and definitions. First we introduce some notations. Suppose that H be a Hilbert space $L^2[a,b]$, where $a,b \in \mathbb{R}$, $\eta_N = \{T_0^*, T_1^*, ..., T_N^*\} \subset H$ and $S = span(\eta_N)$, endowed with the norm

$$\|y_j\|_2 = \left[\int_a^b |y_j(x)|^2\right]^{\frac{1}{2}},\ y_j \in L^2[a,b]. \tag{28}$$

The Hilbert space H with the norm (28) is strictly convex. Therefore, if y_j is an arbitrary element in H, due to S being a finite dimensional vector space, y_j has a unique best approximation out of S such as $s_0 \in S$, such that

$$\|y_j - s_0\|_2 \leq \|y_j - s\|_2, \quad \text{for all s} \in \text{S},$$

where $\|y_j\|_2^2 =< y_j, y_j >$. Since $s_0 \in S$, there exist unique coefficients $\mathbf{A}_j$, $j = 1, ..., k$, such that

$$y_j \simeq s_0 = \sum_{l=0}^{N} a_{j,k} T_k^*(x) = \mathbf{T}^*(x)\mathbf{A}_j.$$

Through these informations, the subsequent theorems, lemmas and definitions can be outlined:

Theorem 3.1. *[20] If H is a normed linear space and W be a finite-dimensional subspace of H, then, given $h \in H$, there exists $w^* \in W$ such that*

$$\|h - w^*\|_2 \leq \|h - w\|_2,\ \textit{for all } h \in W.$$

Definition 3.2. [20] Let $y_j(x)$ be defined on $[a,b]$, the modulus of continuity of $y_j(x)$ on $[a,b]$, $\omega\,(y_j; [a,b]; \delta)$, is defined for $\delta > 0$ by

$$\omega\,(y_j; [a,b]; \delta) = \sup\{|y_j(x_1) - y_j(x_2)| :\ x_1,\, x_2 \in [a,b],\ |x_1 - x_2| < \delta\}.$$

Lemma 3.3. *[20] Suppose that $g(x) = y_j(Ax + B)$ for $c \leq x \leq d$, then*

$$\omega(g; [c,d]; \delta) = \omega(y_j; [Ac + B, Ad + B]; A\delta).$$

Theorem 3.4. *[20] If $f(x)$ is bounded for $0 \leq x \leq 1$, then*

$$\|y_j - P_N(y_j)\|_\infty \leq 3/2\ \omega\left(y_j, [0,1], 1/\sqrt{N}\right), \quad N = 1, 2, ...,$$

where

$$P_N(y_j) = \sum_{k=0}^{N} y_j\left(\frac{k}{N}\right) T_k^*,$$

and $\|y_j\|_\infty = \max\{|y_j(x)| : 0 \leq x \leq 1\}$.

Lemma 3.5. *If $y_j : [a,b] \to \mathbb{R}$ is bounded, $S = span\{T_0^*, T_1^*, ..., T_N^*\}$ and $\mathbf{T}^*(x)\mathbf{A}_j$ is the best approximation to y_j out of S, then*

$$\|y_j - \mathbf{T}^*(x)\mathbf{A}_j\|_2 \leq 3/2\ \left(\sqrt{b-a}\right)\omega\left(y_j, [a,b], (b-a)/\sqrt{N}\right).$$

Proof. Let $x = (b-a)t + a$, then as x varies from a to b, t varies from 0 to 1. Put $y_j(x) = y_j((b-a)t+a)$, then $y_j(x)$ is bounded on $[0,1]$. Using Theorem 3.4 and Lemma 3.3, results in

$$\|y_j - P_N(y_j)\|_\infty \leq 3/2\ \omega\left(y_j, [a,b], (b-a)/\sqrt{N}\right).$$

Since y_j is bounded on $[a,b]$, there exists $M > 0$ such that $|y_j(x)| \leq M$. So,

$$\|y_j\|_2 = \sqrt{\int_a^b |y_j(x)|^2 dx} \leq M\sqrt{b-a},$$

thus, we have $\|y_j\|_2 \leq \sqrt{b-a}\|y_j\|_\infty$. Since $\mathbf{T}^*(x)\mathbf{A}_j$ is the best approximation to y_j out of S and $P_N(y_j) \in S$, it is straightforward to write

$$\begin{aligned}\|y_j - \mathbf{T}^*(x)\mathbf{A}_j\|_2 &\leq \|y_j - B_N(y_j)\|_2 \leq \left(\sqrt{b-a}\right)\|y_j - B_N(y_j)\|_\infty \\ &\leq \frac{3}{2}\left(\sqrt{b-a}\right)\omega\left(y_j, [a,b], \frac{b-a}{\sqrt{N}}\right).\end{aligned}$$

□

Lemma 3.6. *Suppose that $H = L^2[a,b]$ be a real Hilbert space and function $y_j \in H$ is $N+1$ times continuously differentiable, $y_j \in C^{N+1}[a,b]$, and $S = span\{T_0^*, T_1^*, ..., T_N^*\}$. If $\mathbf{T}^*(x)\mathbf{A}_j$ is the best approximation to y_j out of S, then a bound for the absolute error will be obtained by*

$$\|y_j - \mathbf{T}^*(x)\mathbf{A}_j\|_2 \leq \frac{M(b-a)^{N+\frac{3}{2}}}{(N+1)!\sqrt{2N+3}}.$$

Proof. Suppose that $s \in S$, the N-th Taylor polynomial of $y_j(x)$ expanded around $x = a$ is

$$s(x) = \sum_{n=0}^{N} y_j^{(n)}(a)\frac{(x-a)^n}{n!},$$

in addition we have

$$|y_j(x) - s(x)| \leq \left|y_j^{(N+1)}(\xi)\right| \frac{(x-a)^{N+1}}{(N+1)!},$$

where $\xi \in [a,b]$. Since $\mathbf{T}^*(x)\mathbf{A}_j$ is the best approximation to y_j out of S, one obtains

$$\begin{aligned}\|y_j - \mathbf{T}^*(x)\mathbf{A}_j\|_2 &\leq \|y_j - s\|_2 = \sqrt{\int_a^b |f_j(x) - s(x)|^2 dx} \\ &\leq \sqrt{\int_a^b \left(\left|y_j^{(N+1)}(\xi)\right| \frac{(x-a)^{N+1}}{(N+1)!}\right)^2 dx} \leq \sqrt{\frac{M^2(b-a)^{2N+3}}{[(N+1)!]^2(2N+3)}} \\ &= \frac{M(b-a)^{N+\frac{3}{2}}}{(N+1)!\sqrt{2N+3}}.\end{aligned}$$

These complete the proof. □

Due to this error bound, if $N \to \infty$ we have

$$\lim_{N\to\infty} \frac{M(b-a)^{N+\frac{3}{2}}}{(N+1)!\sqrt{2N+3}} = 0,$$

and then, $\|y_j - \mathbf{T}^*(x)\mathbf{A}_j\|_2 \to 0$. These results confirm that the approximate solution $\mathbf{T}^*(x)\mathbf{A}_j$ converges to the exact solution y_j for $j = 1, \ldots, k$.

If we set $\mathbf{K}^f(x,\gamma) = [\kappa^f_{i,j,n}(x,\gamma)]$, $\mathbf{K}^v(x,\gamma) = [\kappa^v_{i,j,n}(x,\gamma)]$, $\mathbf{P}_n(x) = [P_{i,j,n}(x)]$ and $\mathbf{g}(x) = [g_i(x)]$ where $i, j = 1, \ldots, k$, then system (1) can be written in the following matrix form

$$\sum_{n=0}^{n1} \mathbf{P}_n(x)\mathbf{y}^{(n)}(x) = \mathbf{g}(x) + \int_a^b \sum_{n=0}^{n2} \mathbf{K}^f(x,\gamma)\mathbf{y}^{(n)}(\gamma)d\gamma + \int_a^x \sum_{n=0}^{n3} \mathbf{K}^v(x,\gamma)\mathbf{y}^{(n)}(\gamma)d\gamma. \tag{29}$$

Theorem 3.7. *Assume that* ${}^N\mathbf{y}(x)$ *and* $\mathbf{y}(x)$ *are approximate and exact solutions of Eq.* (29) *respectively,* $\mathbf{g}(x)$ *be a function defined on* $[a,b]$, $\mathbf{K}^f(x,\gamma)$ *and* $\mathbf{K}^v(x,\gamma)$ *are sufficiently smooth continuous and arbitrary differentiable kernel functions. Then, we have*

$$\frac{\left\| \sum_{n=0}^{n1} \mathbf{P}_n(x)\left({}^N\mathbf{y}^{(n)}(x) - \mathbf{y}^{(n)}(x)\right) \right\|_\infty}{\|{}^N\mathbf{y}(x) - \mathbf{y}(x)\|_\infty} \leq \frac{n2(n2+1)}{2}\alpha + \frac{n3(n3+1)}{2}\beta, \tag{30}$$

where

$$\alpha = \sup_{a\leq x\leq b} \int_a^b \mathbf{K}^f(x,\gamma)d\gamma, \quad \beta = \sup_{a\leq x\leq b} \int_a^x \mathbf{K}^v(x,\gamma)d\gamma.$$

Proof. *Substitiuting the approximate solution in Eq.* (29) *and some simplifications, results in*

$$\left|\sum_{n=0}^{n1} \mathbf{P}_n(x)\left({}^N\mathbf{y}^{(n)}(x) - \mathbf{y}^{(n)}(x)\right)\right| \leq \int_a^b \sum_{n=0}^{n2} \left|\mathbf{K}^f(x,\gamma)\right| \left|{}^N\mathbf{y}^{(n)}(\gamma) - \mathbf{y}^{(n)}(\gamma)\right| d\gamma + \int_a^x \sum_{n=0}^{n3} |\mathbf{K}^v(x,\gamma)| \left|{}^N\mathbf{y}^{(n)}(\gamma) - \mathbf{y}^{(n)}(\gamma)\right| d\gamma.$$

This implies that

$$\begin{aligned}
&\left\| \sum_{n=0}^{n1} \mathbf{P}_n(x) \left({}^N\mathbf{y}^{(n)}(x) - \mathbf{y}^{(n)}(x) \right) \right\|_\infty \\
&\quad \le \left\| {}^N\mathbf{y}^{(n)}(\gamma) - \mathbf{y}^{(n)}(\gamma) \right\|_\infty \left[\sum_{n=0}^{n2} \int_a^b \left| \mathbf{K}^f(x,\gamma) \right| d\gamma + \sum_{n=0}^{n3} \int_a^x \left| \mathbf{K}^v(x,\gamma) \right| d\gamma \right] \\
&\quad \le \left\| {}^N\mathbf{y}^{(n)}(\gamma) - \mathbf{y}^{(n)}(\gamma) \right\|_\infty \left(\frac{n2(n2+1)}{2}\alpha + \frac{n3(n3+1)}{2}\beta \right),
\end{aligned}$$

which shows that relation (30) holds and the proof is completed. □

4 Numerical experiments

In this section, some numerical examples are presented to justify the efficiency of our method. Moreover, we compare the obtained approximate solution with the results of some other methods. In addition, we will find the actual maximum absolute error as

$$\mathrm{e}_{i,N} = \left\| y_{i,ex}(x) - {}^N y_i(x) \right\|_\infty = max \left\{ \left| y_{i,ex}(x) - {}^N y_i(x) \right|,\ a \le x \le b \right\},\ i = 1, 2, \ldots, k.$$

Example 4.1. Consider the following system of linear FVIDEs

$$\begin{cases} -y_1'(x) + y_2(x) = 1 + x + x^2 + \int_a^x \left(-y_1(\gamma) - y_2(\gamma)\right) d\gamma, \\ y_2'(x) - y_1(x) = -1 - x + \int_a^x \left(-y_1(\gamma) + y_2(\gamma)\right) d\gamma, \end{cases} \tag{31}$$

with the initial conditions $y_1(0) = 1$, $y_2(0) = -1$ and the exact solutions $y_1(x) = x + e^x$, $y_2(x) = x - e^x$. Using Eq. (24) the augmented matrix for the fundamental matrix equation can be written as

$$[\mathbf{W}; \mathbf{G}] = \begin{bmatrix}
1/2 & 11/6 & -1/6 & 21/10 & 3/2 & -1/6 & -1/2 & 1/10 & 11/6 \\
1/6 & 0 & 77/30 & 0 & 1/6 & 1/3 & -1/10 & -1/5 & 1/3 \\
-1/6 & -17/30 & 1/18 & 737/210 & -1/2 & 1/10 & 47/90 & -19/210 & -17/3 \\
-1/10 & 0 & -317/210 & 0 & -1/10 & -1/5 & 19/210 & 17/35 & -1/5 \\
-1/2 & -1/6 & 1/6 & 1/10 & -1/2 & 13/6 & 1/6 & 19/10 & -3/2 \\
1/6 & -1/3 & -1/10 & 1/5 & -1/6 & 0 & 83/30 & 0 & -1/6 \\
1/6 & 1/10 & -37/90 & -19/210 & 1/6 & -23/30 & -1/18 & 155/42 & 1/2 \\
-1/10 & 1/5 & -19/210 & -17/35 & 1/10 & 0 & -71/42 & 0 & 1/10
\end{bmatrix}.$$

From Eq. (25), the matrix form for initial conditions is

$$[\mathbf{U}, \mu] = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 \end{bmatrix}.$$

Thus, the new augmented matrix based on system (27) can be obtained as follows

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]$$
$$=\begin{bmatrix} 1/2 & 11/6 & -1/6 & 21/10 & 3/2 & -1/6 & -1/2 & 1/10 & 11/6 \\ 1/6 & 0 & 77/30 & 0 & 1/6 & 1/3 & -1/10 & -1/5 & 1/3 \\ -1/6 & -17/30 & 1/18 & 737/210 & -1/2 & 1/10 & 47/90 & -19/210 & -17/3 \\ -1/10 & 0 & -317/210 & 0 & -1/10 & -1/5 & 19/210 & 17/35 & -1/5 \\ -1/2 & -1/6 & 1/6 & 1/10 & -1/2 & 13/6 & 1/6 & 19/10 & -3/2 \\ 1/6 & -1/3 & -1/10 & 1/5 & -1/6 & 0 & 83/30 & 0 & -1/6 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 \end{bmatrix}.$$

This linear system gives

$$\mathbf{A}_1=\begin{bmatrix}\frac{753436279}{334342616} & \frac{1805906557}{1337370464} & \frac{35327115}{334342616} & \frac{11776555}{1337370464}\end{bmatrix}^T,$$
$$\mathbf{A}_2=\begin{bmatrix}-\frac{838209151}{668685232} & -\frac{234355939}{668685232} & -\frac{70634565}{668685232} & -\frac{5802545}{668685232}\end{bmatrix}^T.$$

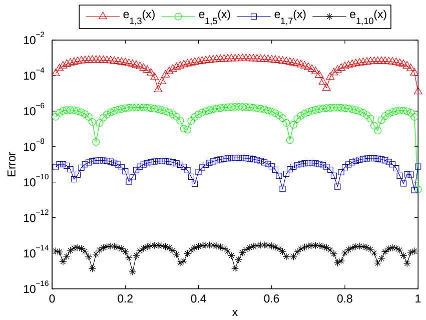
Substituting these elements in (5), result in

$$^{3}y_1(x)=1+2.01389x+0.422615x^2+0.28178x^3,$$
$$^{3}y_2(x)=-1-0.012085x-0.42853x^2-0.27768x^3.$$

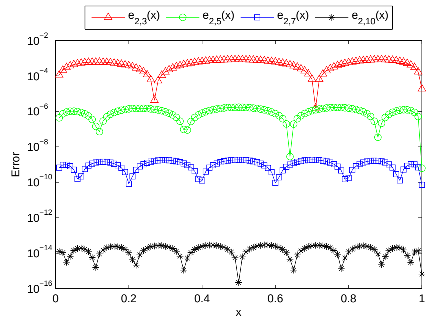
We compare the absolute error functions for different values of N in Figs. 1. In Tables 1 and 2, we list the results obtained by our method together with the Bessel collocation method [28]. Tables 3 and 4 compare the absolute error of this method with the method in [28]. The displayed results show that our method is more accurate than the Bessel collocation method. This figures and tables confirm that the presented method is able to approximate the solution of Eq. (31) with high a accuracy.

Table 1: Comparison of the absolute errors for $N = 7$, 10 of $y_1(x)$ in example (4.1).

	ADM [7]	HPM [24]	Method [28]		Present method	
x_i	$\lvert e_{1,7}(x)\rvert$	$\lvert e_{1,7}(x)\rvert$	$\lvert e_{1,7}(x)\rvert$	$\lvert e_{1,10}(x)\rvert$	$\lvert e_{1,7}(x)\rvert$	$\lvert e_{1,10}(x)\rvert$
0	0	0	0	0	0	0
0.2	3.0e−09	3.0e−09	3.0530e−09	1.1569e−13	4.0901e−10	1.2179e−14
0.4	3.20e−07	3.20e−07	3.2758e−09	1.6609e−13	3.7285e−10	2.4096e−14
0.6	5.364e−06	5.364e−06	3.1170e−09	2.3115e−13	7.7633e−10	2.5423e−14
0.8	3.909e−05	3.909e−05	2.4377e−09	2.9798e−13	6.7835e−10	1.0266e−14
1	1.7986e−04	1.7986e−04	8.4851e−08	4.011e−12	7.5014e−11	4.4179e−16



$y_1(x)$



$y_2(x)$

Figure 1: Comparison of the absolute error functions for system (31) in example (4.1).

Table 2: Comparison of the absolute errors for $N = 7$, 10 of $y_2(x)$ in example (4.1).

	ADM [7]	HPM [24]	Method [28]		Present method	
x_i	$\lvert e_{2,7}(x)\rvert$	$\lvert e_{2,7}(x)\rvert$	$\lvert e_{2,7}(x)\rvert$	$\lvert e_{2,10}(x)\rvert$	$\lvert e_{2,7}(x)\rvert$	$\lvert e_{2,10}(x)\rvert$
0	0	0	0	0	0	0
0.2	2.0e−09	2.0e−09	2.06810e−09	5.3069e−14	4.1272e−10	2.1809e−15
0.4	3.20e−07	3.20e−07	1.6889e−09	2.6645e−15	3.0724e−10	5.0023e−16
0.6	5.359e−06	5.359e−06	1.1459e−09	6.7946e−14	8.0102e−10	2.4265e−16
0.8	3.9028e−05	3.9028e−05	2.0071e−10	1.6120e−13	4.7211e−11	1.0653e−14
1	1.79279e−04	1.79279e−04	8.8759e−08	4.8770e−12	5.2146e−10	2.2325e−15

Table 3: The actual maximum error $|e_{1,N}|$ for system (31) in example (4.1).

N	3	5	7	8	10	12
Our method	9.0605e−04	1.7354e−06	2.3302e−09	5.1536e−11	2.9185e−14	6.1062e−16
Method [28]	7.1428e−03	3.3569e−05	3.3569e−05	3.2950e−09	7.9e−10	7.9e−10

Table 4: The actual maximum error $|e_{2,N}|$ for system (31) in example (4.1).

N	3	5	7	8	10	12
Our method	8.3527e−04	1.7096e−06	1.8470e−09	5.1753e−11	2.9282e−14	4.996e−16
Method [28]	7.7408e−03	3.5526e−05	8.8769e−08	3.5872e−09	8.0e−10	1.0e−11

Example 4.2. Let

$$\begin{aligned} g_1(x) &= 3\sin(x)(x-1) + \cos(x)(2 - 2cos(1) - 2sin(1) + 0.5x) \\ &\quad - 0.05x^2\sin^2(1) - 0.5\sin(x)cos^2(x), \\ g_2(x) &= -\cos(x)(x + sin(1)cos(1)) + 0.5x^2 sin^2(1) - \sin(x)(1 + 0.5x) \\ &\quad - 0.5\sin^2(x)\cos(x) + x, \end{aligned}$$

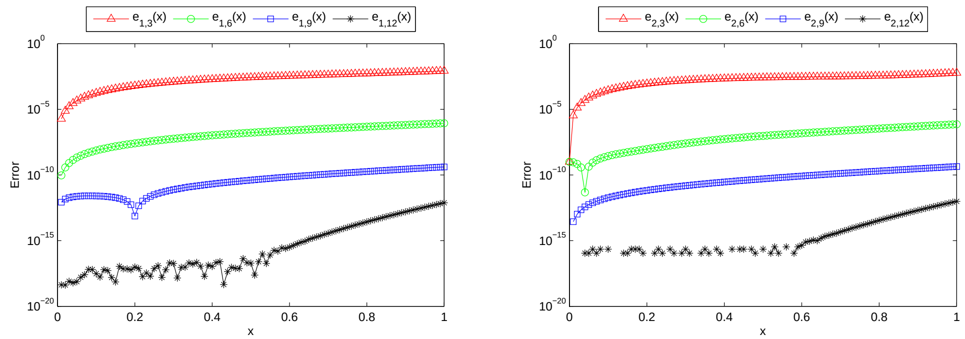
and consider the following system of linear FVIDEs

$$\begin{cases} y_1''(x) - 3xy_2'(x) - 2y_1(x) = g_1(x) + \int_a^b k_1(x,\gamma)d\gamma + \int_a^x k_2(x,\gamma)d\gamma, \\ y_2''(x) - 2xy_1'(x) + xy_2(x) = g_2(x) + \int_a^b k_3(x,\gamma)d\gamma + \int_a^x k_4(x,\gamma)d\gamma, \end{cases} \tag{32}$$

where $k_1(x,\gamma) = 2\gamma\cos(x)y_1'(\gamma) - x^2\sin(\gamma)y_2''(\gamma)$, $k_2(x,\gamma) = x\cos(\gamma)y_1(\gamma) - xsin(\gamma)y_1'(\gamma) + \cos(x)\sin(\gamma)y_2'(\gamma)$, $k_3(x,\gamma) = x^2\cos(\gamma)y_1''(\gamma) + 2\cos(x)$ $\sin(\gamma)y_2'(\gamma)$ and $k_4(x,\gamma) = \sin(x)\cos(\gamma)y_1'(\gamma) - \gamma\cos(x)y_2'(\gamma) + \gamma\sin(x)y_2''(\gamma)$ with the initial conditions $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$ and the exact solutions $y_1(x) = \sin(x)$, $y_2(x) = \cos(x)$. Applying the presented scheme, we obtain the approximate solutions of Eq. (32) for different values of N. We compare the absolute error functions for different values of N in Figs. 2. The results of the proposed method for this example are exhibited in Table 5 and 6 with various choices of N. In Table 7, 8, the actual maximum absolute errors are compared with the method in [28]. One can observe that, our method is more effective, because for the same N, it obtains better results.

Table 5: Comparison of the absolute errors for N = 3, 6, 9 and 12 for $y_1(x)$ in system (32).

	N			
x_i	3	6	9	12
0	0	0	0	0
0.2	6.1960e−04	2.5636e−08	7.4218e−14	2.5054e−14
0.4	1.9490e−03	1.0421e−07	2.0634e−11	3.2062e−14
0.6	3.4779e−03	2.4438e−07	7.1539e−11	3.5908e−13
0.8	5.3152e−03	4.7406e−07	1.8320e−10	2.6843e−13
1	8.4668e−03	4.7406e−07	4.1209e−10	3.5147e−13



$y_1(x)$ $y_2(x)$

Figure 2: Comparison of the absolute error functions for system (32) in example (4.2)

Table 6: Comparison of the absolute errors for $N = 3$, 6, 9 and 12 for $y_2(x)$ in system (32).

	N			
x_i	3	6	9	12
0	0	0	0	0
0.2	8.9762e−04	9.3542e−09	7.9332e−11	3.3406e−14
0.4	2.1817e−03	5.4030e−08	7.7096e−10	1.0844e−14
0.6	2.8996e−03	1.5024e−07	7.1717e−10	4.7037e−14
0.8	3.5623e−03	3.4352e−07	6.0038e−10	3.4863e−13
1	5.9927e−03	7.3350e−07	3.5963e−10	8.8140e−13

Table 7: The actual maximum error $|e_{1,N}|$ for system (32) in example (4.2).

N	3	6	9	12	14
Our method	8.4668e−03	8.7364e−07	5.1781e−10	7.6195e−13	5.3674e−16
Method [28]	5.8286e−02	5.3095e−05	8.3560e−09	8.3560e−09	3.1242e−13

Example 4.3. Consider the following nonlinear Fredholm-Volterra integro-

Table 8: The actual maximum error $|e_{2,N}|$ for system (32) in example (4.2).

N	3	6	9	12	14
Our method	5.993e−03	5.993e−03	1.408e−10	9.8133e−13	1.0152e−16
Method [28]	7.0965e−02	2.4116e−05	3.0543e−09	3.4644e−11	2.6368e−13

differential equation

$$y'(x) + 2xy(x) = g(x) + \int_0^1 (x-t)y(t)dt + \int_0^x (x+t)[y(t)]^3 dt, \quad (33)$$

where $g(x) = \left(-\frac{2}{3}x + \frac{1}{9}\right)e^{3x} + (2x+1)e^x + \left(\frac{4}{3} - e\right)x + \frac{8}{9}$ and the initial condition $y(0) = 1$. The exact solution is $y(x) = e^x$. By applying our method, the approximate solutions of the problem for various values of N will be obtain as follows

$$^2y(x) = 0.83990x^2 + 0.87007x + 1, \ ^3y(x) = 0.27871x^3 + 0.42592x^2 + 1.01316x + 1,$$
$$^4y(x) = 0.06953x^4 + 0.14057x^3 + 0.50911x^2 + 0.99906x + 1, \ldots .$$

One can see that by increasing the amount of N, the Taylor series expansion of e^x will be achieved. Hence, the results of absolute error function for $N = 3$, 4, 7 and 9 are depicted in Fig. 3. In Table 9, we tabulate the maximum absolute errors for $N = 7$, 10 and 26. In this Table, our results are also compared with the results of the reproducing kernel Hilbert space (RKHS)[2] , the Taylor [16] and the collocation methods [26]. It is observe that the results obtained by the presented method are better than those of other methods. Moreover, these results confirms that with increasing the amount of N, the error will be decreased.

Table 9: Comparison of the actual maximum error $|e_N|$ for system (33).

x	Method in [2]	Method in [16]	Method in [26]	Our method	
	N=26, n=1	N=7	N=7	N=7	N=9
0	0	0	0	0	0
0.2	1.51514e−06	1.8156e−09	3.4184e−07	6.1619e−10	1.6838e−14
0.4	3.85161e−06	1.3355e−09	6.0236e−07	3.1782e−12	9.8516e−15
0.6	7.06648e−06	1.0530e−09	8.9961e−07	8.5397e−11	1.8006e−14
0.8	1.13514e−05	1.0261e−09	1.0715e−06	6.2825e−10	1.6482e−14
1	1.72540e−05	9.1675e−08	4.2846e−07	1.4178e−09	1.5138e−14

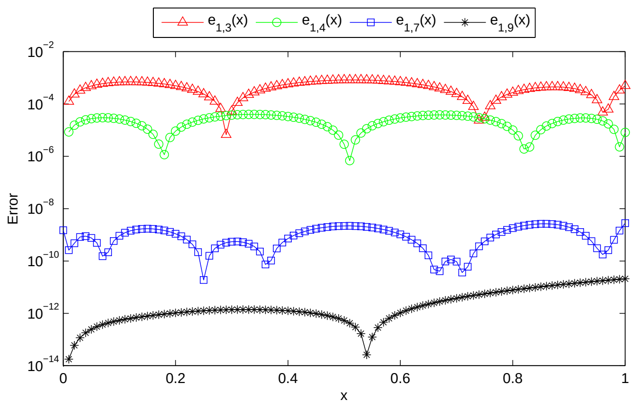


Figure 3: Comparison of the absolute error function for system (33)

Example 4.4. [11, 12] Consider the following nonlinear system of Volterra-Fredholm integral equations

$$\begin{cases} y_1(x) = g_1(x) + \int_{-1}^{x} (t^2 - x) y_1(t) dt + \int_{-1}^{1} \left(xt^2 y_1(t) + x(t+1) y_2^2(t) \right) dt, \\ y_2(x) = g_2(x) + \int_{-1}^{x} 2y_2(t) + \int_{-1}^{1} 3xy_1^2(t) dt, \end{cases}$$

where $g_1(x) = -\frac{x^4}{4} + \frac{5}{6}x^3 - x^2 - \frac{x}{10} - \frac{5}{12}$ and $g_2(x) = -\frac{2}{3}x^3 + 2x^2 - 9x - \frac{5}{3}$. By choosing $N = 2$ and applying the proposed approach for this problem, the fundamental system of equations will be built. With solving this nonlinear algebraic system, one obtain

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T, \quad \mathbf{A}_2 = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}^T.$$

Substituting these matrices in Eq. (5), the numerical solutions will be determined as ${}^2y_1(x) = x - 1$ and ${}^2y_2(x) = x^2 - x$, which coincide with the exact solution. In references [11] and [12] the authors obtained approximate solution for this system by using Chebyshev collocation, Bernstein polynomials and the hybrid Bernstein Block-Pulse functions methods, whereas, we obtained the exact solution.

5 Conclusion

In this paper, we have applied the shifted Chebyshev Galerkin technique for the numerical solution of systems of linear and nonlinear FVIDEs which includes the derivatives of the unknown functions in integral parts. Furthermore, we have presented error estimation for the mentioned systems. Mathematical analysis about the numerical errors and convergence were also discussed. To show the performance of error estimation for the proposed method, we applied the presented scheme to several examples. The numerical results showed that the method employed in this work was valid and could be used as a very accurate algorithm for solving the linear and nonlinear FVIDEs. Moreover, these satisfactory results revealed that the proposed method was more powerful with respect to some other well-known methods. In addition, this method should be developed with some modifications for solving systems of partial differential equations.

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