Module Amenability and Tensor Product of Semigroup Algebras

A. Bodaghi
Islamic Azad University-Garmsar Branch

Abstract. Let $S$ be an inverse semigroup with an upward directed set of idempotents $E$. In this paper we prove that if $S$ is amenable, then $\ell^1(S) \hat{\otimes} \ell^1(S)$ is module amenable as an $\ell^1(E)$-module. Also we show that $\ell^1(S) \hat{\otimes} \ell^1(S)$ is module super-amenable if an appropriate group homomorphic image of $S$ is finite.

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1. Introduction

The notion of amenability of Banach algebras was introduced by Barry Johnson in [9]. A Banach algebra $A$ is amenable if every bounded derivation from $A$ into any dual Banach $A$-module is inner, equivalently if $H^1(A, X^*) = \{0\}$ for every Banach $A$-module $X$, where $H^1(A, X^*)$ is the first Hochschild cohomology group of $A$ with coefficients in $X^*$. He proved in [9, Proposition 5.4] that if $A$ and $B$ are amenable Banach algebra, then so is $A \hat{\otimes} B$ (see also [6, Corollary 2.9.62]). Also $A$ is called super-amenable (contractible) if $H^1(A, X) = \{0\}$ for every Banach $A$-bimodule $X$ (see [6,12]). It is known $A \hat{\otimes} B$ is super-amenable if $A$ and $B$ are super-amenable [12, Exercise 4.1.4].

For a discrete semigroup $S$, $\ell^\infty(S)$ is the Banach algebra of bounded complex-valued functions on $S$ with the supremum norm and pointwise
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multiplication. For each \( t \in S \) and \( f \in \ell^\infty(S) \), let \( L_t f \) and \( R_t f \) denote the left and the right translations of \( f \) by \( t \), that is \( \langle L_t f, s \rangle = \langle f, ts \rangle \) and \( \langle R_t f, s \rangle = \langle f, st \rangle \), for each \( s \in S \). Then a linear functional \( m \in (\ell^\infty(S))^* \) is called a mean if \( \|m\| = \langle m, 1 \rangle = 1 \); \( m \) is called a left (right) invariant mean if \( \langle m, L_t f \rangle = \langle m, f \rangle \) (\( \langle m, R_t f \rangle = \langle m, f \rangle \), respectively) for all \( s \in S \) and \( f \in \ell^\infty(S) \). A discrete semigroup \( S \) is called amenable if there exists a mean \( m \) on \( \ell^\infty(S) \) which is both left and right invariant (see [7]). An inverse semigroup is a discrete semigroup \( S \) such that for each \( s \in S \), there is a unique element \( s^* \in S \) with \( ss^* s = s \) and \( s^* ss^* = s^* \). Elements of the form \( ss^* \) are called idempotents of \( S \). For an inverse semigroup \( S \), a left invariant mean on \( \ell^\infty(S) \) is right invariant and vise versa.

M. Amini in [1] introduced the concept of module amenability for a Banach algebra. He showed that for an inverse semigroup \( S \) with set of idempotents \( E \), the semigroup algebra \( \ell^1(S) \) is \( \ell^1(E) \)-module amenable if and only if \( S \) is amenable.

This extends the Johnson’s theorem [9, Theorem 2.5] in the discrete case) which asserts that for a discrete group \( G \), \( \ell^1(G) \) is amenable if and only if \( G \) is amenable. The author and Amini in [4] introduced the concept of module super-amenability and showed that for an inverse semigroup \( S \), the semigroup algebra \( \ell^1(S) \) is module super-amenable if and only if the group homomorphic image \( S/\approx \) of \( S \) is finite, where \( \approx \) is an equivalence relation on \( S \).

In part two of this paper, we show that when \( \mathfrak{A} \) acts trivially on \( \mathcal{A} \) from left then under some mild conditions, module amenability of \( \mathcal{A} \otimes \mathcal{A} \) implies amenability of \( \mathcal{A}/J \otimes \mathcal{A}/J \) and vise versa, where \( J \) is the closed ideal of \( \mathcal{A} \) generated by \( \alpha \cdot (ab) - (ab) \cdot \alpha \) for all \( a \in \mathcal{A} \) and \( \alpha \in \mathfrak{A} \). There is a similar result for super amenability.

Finally, we prove that if \( S \) is an amenable inverse semigroup with an upward directed set of idempotents \( E \), then \( \ell^1(S) \otimes \ell^1(S) \) is module amenable as an \( \ell^1(E) \)-module. Also we show that \( \ell^1(S) \otimes \ell^1(S) \) is module super-amenable when the appropriate group homomorphic image \( S/\approx \) is finite.
2. Module Amenability of the Tensor Product of Banach Algebras

Let \( \mathcal{A} \) and \( \mathfrak{A} \) be Banach algebras such that \( \mathcal{A} \) is a Banach \( \mathfrak{A} \)-bimodule with compatible actions, as follows

\[
\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).
\]

Let \( X \) be a Banach \( \mathcal{A} \)-bimodule and a Banach \( \mathfrak{A} \)-bimodule with compatible actions, that is

\[
\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot x) \cdot \alpha = \alpha \cdot (x \cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X)
\]

and the same for the right or two-sided actions. Then we say that \( X \) is a Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module. If moreover \( \alpha \cdot x = x \cdot \alpha \) for all \( \alpha \in \mathfrak{A}, x \in X \), then \( X \) is called a commutative \( \mathcal{A} \)-\( \mathfrak{A} \)-module. If \( X \) is a commutative Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module, then so is \( X^* \), the first dual space of \( X \), where the actions of \( \mathcal{A} \) and \( \mathfrak{A} \) on \( X^* \) are defined as follows

\[
\langle \alpha \cdot f, x \rangle = \langle f, x \cdot \alpha \rangle, \quad \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X, f \in X^*)
\]

and the same for the right actions.

Note that, in general, \( \mathcal{A} \) is not an \( \mathcal{A} \)-\( \mathfrak{A} \)-module because \( \mathcal{A} \) does not satisfy in the compatibility condition \( a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \) for all \( \alpha \in \mathfrak{A}, a, b \in \mathcal{A} \) [2]. But when \( \mathcal{A} \) is a commutative \( \mathfrak{A} \)-module and acts on itself by multiplication from both sides, then it is also a Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module.

It is well known that \( \mathcal{A} \# \mathcal{A} \), the projective tensor product of \( \mathcal{A} \) and \( \mathcal{A} \) is a Banach algebra with respect to the canonical multiplication defined by \( (a \otimes b)(c \otimes d) = (ac \otimes bd) \). Also it is a Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-bimodule and a Banach \( \mathfrak{A} \)-bimodule by the following usual actions:

\[
\alpha \cdot (a \otimes b) = (\alpha \cdot a) \otimes b, \quad c \cdot (a \otimes b) = (ca) \otimes b \quad (\alpha \in \mathfrak{A}, a, b, c \in \mathcal{A}),
\]

Similarly, for the right actions consider the module projective tensor product \( \mathcal{A} \#_{\mathfrak{A}} \mathcal{A} \) which is isomorphic to the quotient space \( (\mathcal{A} \# \mathcal{A})/I \), where \( I \) is the closed ideal of the projective tensor product \( \mathcal{A} \# \mathcal{A} \) generated by elements of the form \( \alpha \cdot a \otimes b - a \otimes b \cdot \alpha \) for \( \alpha \in \mathfrak{A}, a, b \in \mathcal{A} \) [11]. Also we consider \( J \), the closed ideal of \( \mathcal{A} \) generated by elements
of the form \((\alpha \cdot a)b - a(b \cdot \alpha)\) for \(\alpha \in \mathfrak{A}, a, b \in \mathcal{A}\). Then \(\mathcal{A}/J\) is Banach \(\mathcal{A}\)-\(\mathfrak{A}\)-module when \(\mathcal{A}\) acts on \(\mathcal{A}/J\) canonically.

Let \(\mathcal{A}\) and \(\mathfrak{A}\) be as in the above and \(X\) be a Banach \(\mathcal{A}\)-\(\mathfrak{A}\)-module. A bounded map \(D : \mathcal{A} \rightarrow X\) is called a module derivation if

\[
D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}),
\]

and

\[
D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).
\]

Although \(D\) is not necessary linear, but still its boundedness implies its norm continuity (since it preserves subtraction). When \(X\) is commutative \(\mathcal{A}\)-\(\mathfrak{A}\)-module, each \(x \in X\) defines a module derivation

\[
D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).
\]

These are called inner module derivations. The Banach algebra \(\mathcal{A}\) is called module amenable (as an \(\mathfrak{A}\)-module) if for any commutative Banach \(\mathcal{A}\)-\(\mathfrak{A}\)-module \(X\), each module derivation \(D : \mathcal{A} \rightarrow X^*\) is inner [1]. Similarly, \(\mathcal{A}\) is called module super-amenable if each module derivation \(D : \mathcal{A} \rightarrow X\) is inner [4].

We say the Banach algebra \(\mathfrak{A}\) acts trivially on \(\mathcal{A}\) from left (right) if for each \(\alpha \in \mathfrak{A}\) and \(a \in \mathcal{A}\), \(\alpha \cdot a = f(\alpha)a \ (a \cdot \alpha = f(\alpha)a)\), where \(f\) is a continuous linear functional on \(\mathfrak{A}\). The following lemma is proved in [3].

**Lemma 2.1.** Let \(\mathcal{A}\) be a Banach algebra and Banach \(\mathfrak{A}\)-module with compatible actions, and \(J_0\) be a closed ideal of \(\mathcal{A}\) such that \(J \subseteq J_0\). If \(\mathcal{A}/J_0\) has a left or right identity \(e + J_0\), then for each \(\alpha \in \mathfrak{A}\) and \(a \in \mathcal{A}\) we have \(a \cdot \alpha - \alpha \cdot a \in J_0\), i.e., \(\mathcal{A}/J_0\) is commutative Banach \(\mathfrak{A}\)-module.

Recall that \(\mathfrak{A}\) has a bounded approximate identity for \(\mathcal{A}\) if there is a bounded net \(\{\gamma_i\}\) in \(\mathfrak{A}\) such that for each \(a \in \mathcal{A}\), \(\|\gamma_i \cdot a - a\| \rightarrow 0\) and \(\|a \cdot \gamma_i - a\| \rightarrow 0\), as \(i \rightarrow \infty\).

**Theorem 2.2.** Let \(\mathcal{A}\) be a Banach \(\mathfrak{A}\)-module with trivial left action and \(\mathcal{A}/J\) has an identity. If \(\mathcal{A} \otimes \mathfrak{A}\) is module amenable (module super-amenable), then \(\mathcal{A}/J \otimes \mathcal{A}/J\) is amenable (module super-amenable). The converse is true if \(\mathfrak{A}\) has a bounded approximate identity for \(\mathcal{A}\).
Proof. We prove the result for the module amenability. Let $X$ be a unital $A/J \hat{\otimes} A/J$-bimodule and $D : A/J \hat{\otimes} A/J \to X^*$ be a bounded derivation (see [5, Lemma 43.6]). Then $X$ is an $A \hat{\otimes} A$-bimodule with module actions given by

$$(a \otimes b) \cdot x := ((a + J) \otimes (b + J)) \cdot x, \quad x \cdot (a \otimes b) := x \cdot ((a + J) \otimes (b + J)) \quad (x \in X, a \in A),$$

and $X$ is $\mathcal{A}$-bimodule with trivial actions, that is $\alpha \cdot x = x \cdot \alpha = f(\alpha)x$, for each $x \in X$ and $\alpha \in \mathcal{A}$ which $f$ is a continuous linear functional on $\mathcal{A}$. Since $f(\alpha)a - a \cdot \alpha \in J$ (see Lemma 2.1.), we have $f(\alpha)a + J = a \cdot \alpha + J$, for each $\alpha \in \mathcal{A}$, and the actions of $\mathcal{A}$ and $A \hat{\otimes} A$ on $X$ are compatible. Therefore $X$ is commutative Banach $A \hat{\otimes} A$-module. Consider $\Phi : (A \hat{\otimes} A)/I \to A/J \hat{\otimes} A/J$ defined by

$$\Phi((a \otimes b) + I) = (a + J) \otimes (b + J).$$

For each $a, b \in A$ and $\alpha \in \mathcal{A}$ we have

$$(\alpha \cdot a + J) \otimes (b + J) - (a + J) \otimes (b \cdot \alpha + J) = (f(\alpha)a + J) \otimes (b + J) \quad - (a + J) \otimes (f(\alpha)b + J) = f(\alpha)(a + J) \otimes (b + J) \quad - f(\alpha)(a + J) \otimes (b + J) = 0.$$

We have used Lemma 2.1., in the first equality, hence $\Phi$ is well defined. Obviously $\Phi$ is $\mathcal{A}$-bimodule morphism. We show that the map $\overline{D} = D \circ \Phi \circ \pi : A \hat{\otimes} A \to X^*$ is module derivation where $\pi : A \hat{\otimes} A \to (A \hat{\otimes} A)/I$ is the projection map. For each $a, b, c, d \in A$ and $\alpha \in \mathcal{A}$, we have

$$\overline{D}((a \otimes b)(c \otimes d)) = D(((a + J) \otimes (b + J))((c + J) \otimes (d + J))) \quad = D((a + J) \otimes (b + J)) \cdot ((c + J) \otimes (d + J)) \quad + ((a + J) \otimes (b + J)) \cdot D((c + J) \otimes (d + J)) \quad = \overline{D}(a \otimes b) \cdot (c \otimes d) + (a \otimes b) \cdot \overline{D}(c \otimes d).$$

For each $a, b \in A$ we have $\overline{D}((a \otimes b) \pm (c \otimes d)) = \overline{D}(a \otimes b) \pm \overline{D}(c \otimes d)$. 


Also $A/J \hat{\otimes} A/J$ is an $\mathfrak{A}$-bimodule, hence for $\alpha \in \mathfrak{A}$, we have
\[
\overline{D}((a \otimes b) \cdot \alpha) = D((a + J) \otimes (b \cdot \alpha + J)) \\
= D((a + J) \otimes (f(\alpha)b + J)) \\
= f(\alpha)D((a + J) \otimes (b + J)) \\
= \overline{D}(a \otimes b) \cdot \alpha.
\]
On the other hand, since the left $\mathfrak{A}$-module actions on $A$ and $X$ are trivial, $\overline{D}(\alpha \cdot (a \otimes b)) = \overline{D}(f(\alpha)(a \otimes b)) = \alpha \cdot \overline{D}(a \otimes b)$. Therefore there exists $x^* \in X^*$ such that $\overline{D}(a \otimes b) = (a \otimes b) \cdot x^* - x^* \cdot (a \otimes b)$, hence $D((a + J) \otimes (b + J)) = ((a + J) \otimes (b + J)) \cdot x^* - x^* \cdot ((a + J) \otimes (b + J))$, and so $D$ is inner.

For the converse, we note that for every derivation $D : A \rightarrow X$ on unital Banach algebra $A$ with identity $e$, we have $D(e) = 0$ and without loss of generality we can assume that $e \cdot D(a) = D(a) \cdot e = D(a)$ for all $a \in A$. We use this fact in the rest of the proof. Now, suppose that $X$ is a commutative Banach $A \hat{\otimes} A$-$\mathfrak{A}$-module. We consider the following module actions $A/J \hat{\otimes} A/J$ on $X$,
\[
((a + J) \otimes (b + J)) \cdot x := (a \otimes b) \cdot x, \quad x \cdot ((a + J) \otimes (b + J)) := x \cdot (a \otimes b) \quad (x \in X, a \in A).
\]
For each $a, b, c, d \in A$, $x \in X$, and $\alpha, \beta \in \mathfrak{A}$, we have
\[
((\alpha \cdot ab - ab \cdot \alpha) \otimes (\beta \cdot cd - cd \cdot \beta)) \cdot x = (\alpha \cdot ab \otimes \beta \cdot cd - \alpha \cdot ab \otimes cd \cdot \beta \\
- \alpha \cdot ab \otimes \beta \cdot cd \\
+ ab \cdot \alpha \otimes cd \cdot \beta) \cdot x \\
= \beta \cdot ((f(\alpha)ab \otimes cd) \cdot x) \\
- ((f(\alpha)ab \otimes cd) \cdot x) \cdot \beta \\
- \beta \cdot ((ab \cdot \alpha \otimes cd) \cdot x) \\
+ ((ab \cdot \alpha \otimes cd) \cdot x) \cdot \beta = 0.
\]
Similarly if $a \in J$ or $b \in J$, we can show that $(a \otimes b) \cdot x = 0$ and $x \cdot (a \otimes b) = 0$. Therefore $X$ is a Banach $A/J \hat{\otimes} A/J$-bimodule. Suppose that $D : A \hat{\otimes} A \rightarrow X^*$ is a module derivation, and consider $\tilde{D} : A/J \hat{\otimes} A/J \rightarrow X^*$ defined by $\tilde{D}((a + J) \otimes (b + J)) := D(a \otimes b)$, for all $a, b \in A$. Suppose
that $e + J$ is identity for $A/J$, we have

$$D(a \otimes (a \cdot cd - cd \cdot e)) = \alpha \cdot D(a \otimes cd) - D(a \otimes cd) \cdot \alpha$$

$$= \alpha \cdot D(ae \otimes cd) - D(ae \otimes cd) \cdot \alpha$$

$$= \alpha \cdot D(ae \otimes c) \cdot (e \otimes d) + \alpha \cdot (a \otimes c) \cdot D(e \otimes d)$$

$$- D(a \otimes c) \cdot (e \otimes d) \cdot \alpha - (a \otimes c) \cdot D(e \otimes d) \cdot \alpha = 0.$$  

Although $ae$ is not equal with $a$, but we have

$$D(a \otimes cd) = \tilde{D}((a+J) \otimes (cd+J)) = \tilde{D}((ae+J) \otimes (cd+J)) = D(ae \otimes cd).$$

By the above observation, $\tilde{D}$ is also well-defined. Suppose that $A$ has a bounded approximate identity $(\gamma_i)$ for $A$. Since $f$ is bounded, \[\{ |f(\gamma_i)| \}\] is a bounded sequence in $\mathbb{C}$. Without loss of generality, we may assume that $f(\gamma_i) \to 1$, as $i \to \infty$. Then for each $\lambda \in \mathbb{C}$ we have

$$e \cdot (\lambda \gamma_i) - f(\gamma_i)e = (\lambda e) \cdot \gamma_i - f(\gamma_i)e \to \lambda e - e$$

in norm. Since $J$ is a closed ideal of $A$, $\lambda e - e \in J$. Next, for each $\lambda \in \mathbb{C}$, and $a, b \in A$, we have

$$\tilde{D}((\lambda a + J) \otimes (b + J)) = \tilde{D}((a + J) \otimes (b + J))(e + J) \otimes (\lambda e + J)$$

$$= \tilde{D}((a + J) \otimes (b + J)) \cdot ((e + J) \otimes (\lambda e + J))$$

$$+ ((a + J) \otimes (b + J)) \cdot \tilde{D}((e + J) \otimes (\lambda e + J))$$

$$= \lambda \tilde{D}((a + J) \otimes (b + J)) \cdot ((e + J) \otimes (e + J))$$

$$+ ((a + J) \otimes (b + J)) \cdot \tilde{D}((e + J) \otimes (e + J))$$

$$= \lambda \tilde{D}((a + J) \otimes (b + J)).$$

Thus $\tilde{D}$ is $\mathbb{C}$-linear, and so it is inner. Therefore $D$ is an inner module derivation. □

In this part we find conditions on a (discrete) inverse semigroup $S$ such that the tensor product $\ell^1(S) \hat{\otimes} \ell^1(S)$ is $\ell^1(E)$-module amenable and super-amenable, where $E$ is the set of idempotents of $S$, acting on $S$ trivially from left and by multiplication from right. Let $S$ be an inverse semigroup with set idempotent $E$, where the order of $E$ is defined by

$$e \preceq d \iff ed = e \quad (e, d \in E).$$
It is easy to show that $E$ is a (commutative) subsemigroup of $S$ [8, Theorem V.1.2]. In particular $\ell^1(E)$ could be regard as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions ([1]). Here we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \cdot \delta_e = \delta_{se} \cdot \delta_e \quad (s, e \in E).$$

In this case, the ideal $J$ is the closed linear span of $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E\}$. We consider an equivalence relation on $S$ as follows

$$s \approx t \iff \delta_s - \delta_t \in J \quad (s, t \in S).$$

Recall that $E$ is called upward directed if for every $e, f \in E$ there exists $g \in E$ such that $eg = e$ and $fg = f$. This is precisely the assertion that $S$ satisfies the $D_1$ condition of Duncan and Namioka [7]. It is shown in [10, Theorem 3.2.], that if $E$ is upward directed, then the quotient $S/\approx$ is a discrete group. As in [10, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$. With the above notations, $\ell^1(S)/J \cong \ell^1(S/\approx)$ is a commutative $\ell^1(E)$-bimodule with the following actions

$$\delta_e \cdot (\delta_s + J) = \delta_s + J, \quad (\delta_s + J) \cdot \delta_e = \delta_{se} + J \quad (s, e \in E).$$

**Theorem 2.3.** Let $S$ be an inverse semigroup with an upward directed set of idempotents $E$ and $\ell^1(S)$ be a Banach $\ell^1(E)$-module with trivial left action and canonical right action. Then the following statements hold:

(i) If $S$ is amenable, then $\ell^1(S) \hat{\otimes} \ell^1(S)$ is module amenable.

(ii) If $S/\approx$ is finite, then $\ell^1(S) \hat{\otimes} \ell^1(S)$ is module super-amenable.

**Proof.** (i) The semigroup algebra $S$ is amenable if and only if $\ell^1(S)$ is module amenable [1, Theorem 3.1]. Thus $\ell^1(S/\approx)$ is unital amenable Banach algebra by [3, Proposition 3.2], and so the tensor product $\ell^1(S/\approx) \hat{\otimes} \ell^1(S/\approx)$ is amenable [6, Corollary 2.9.62]. Now the proof is completed by using Theorem 2.2.

(ii) Since $S/\approx$ is a finite (discrete)group, $\ell^1(S)$ is module super-amenable as $\ell^1(E)$-module, hence $\ell^1(S/\approx)$ is super-amenable by [4,
Lemma 2.7. By [12, Exercise 4.1.4], $\ell^1(S/ \approx) \hat{\otimes} \ell^1(S/ \approx)$ is super-amenable. Now the result follows from Theorem 2.2 with $A = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$. □

Example 2.4. (i) Let $C$ be the bicyclic inverse semigroup generated by $a$ and $b$, that is

$$C = \{a^mb^n : m, n \geq 0\}, \quad (a^mb^n)^* = a^nb^m.$$  

The set of idempotents of $C$ is $E_C = \{a^mb^n : n = 0, 1, \ldots\}$ which is totally ordered (and so is upward directed) with the following order

$$a^nb^n \preceq a^mb^m \iff m \preceq n.$$  

It is shown in [3] that $C/ \approx$ is isomorphic to the group of integers $\mathbb{Z}$, hence $C$ is amenable. Therefore the tensor product $\ell^1(C) \hat{\otimes} \ell^1(C)$ is module amenable by Theorem 2.3.

(ii) Let $(\mathbb{N}, \lor)$ be the commutative semigroup of positive integers with maximum operation $m \lor n = \max(m, n)$, then each element of $\mathbb{N}$ is an idempotent, that is $E_{\mathbb{N}} = \mathbb{N}$. Hence $\mathbb{N}/\approx$ is the trivial group with one element. Therefore by Theorem 2.2., the tensor product $\ell^1(\mathbb{N}) \hat{\otimes} \ell^1(\mathbb{N})$ is module super-amenable, as an $\ell^1(\mathbb{N})$-module.

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References


**Abasalt Bodaghi**

Department of Mathematics  
Assistant Professor of Mathematics  
Islamic Azad University, Garmsar-Branch  
Garmsar, Iran.  
E-mail: abasalt.bodaghi@gmail.com