Eigenfunctions of the 
Weighted Composition Operators

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Abstract. In the present paper, we characterize the eigenfunctions of a weighted composition operator on space of holomorphic function on the unit disk.

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1. Introduction

A weighted composition operator $C_{\varphi,\psi}$ is an operator that maps $f \in H(U)$, the space of holomorphic functions on the unit disk $U$, into $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$, where $\varphi$ and $\psi$ are analytic functions defined in $U$ such that $\psi(U) \subseteq U$. When $\varphi \equiv 1$, we just have the composition operator $C_{\psi}$ defined by $C_{\psi}(f) = f \circ \psi$.

The eigenfunctions of a composition operator on the classical Hardy space $H^2$, induced by a hyperbolic disk automorphism, are considered in [2, 4, 5] where it has been shown that many eigenfunctions of a composition operator can be found in the doubly cyclic subspace generated by special functions in $H^2$.

Studying the eigenfunctions of weighted composition operators entails a study of the iterate behavior of holomorphic self maps. The holomorphic self maps of $U$ are divided into classes of elliptic and non-elliptic type. The elliptic type is an automorphism and has a fixed point in $U$. It is
well known that this map is conjugate to a rotation \( z \rightarrow \lambda z \) for some complex number \( \lambda \) with \( |\lambda| = 1 \). The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem \([1, 6, 7]\). In the following notation " \( n \xrightarrow{k} \) " means uniformly converges on compact subsets of \( \mathbb{U} \) and \( \psi_n \) denotes the composition of \( \psi \) with itself \( n \)-times.

**Denjoy-Wolff Iteration Theorem.** Suppose \( \psi \) is a holomorphic self-map of \( \mathbb{U} \) that is not an elliptic automorphism. Then

(i) If \( \psi \) has a fixed point \( w \in \mathbb{U} \), then \( \psi_n \xrightarrow{k} w \) and \( |\psi'(w)| < 1 \).

(ii) If \( \psi \) has no fixed point in \( \mathbb{U} \), then there is a point \( w \in \partial \mathbb{U} \) such that \( \psi_n \xrightarrow{k} w \) and the angular derivative of \( \psi \) exists at \( w \), with \( 0 < \psi'(w) \leq 1 \).

We call the unique attracting point \( w \), the Denjoy-Wolff point of \( \psi \). By the Denjoy-Wolff Iteration Theorem, a general classification of a non-elliptic holomorphic self maps of \( \mathbb{U} \) can be given: let \( w \) be the Denjoy-Wolff point of a holomorphic self-map of \( \mathbb{U} \). We say \( \psi \) is of dilation type if \( w \in \mathbb{U} \), of hyperbolic type if \( w \in \partial \mathbb{U} \) and \( \psi'(w) < 1 \), and of parabolic type if \( w \in \partial \mathbb{U} \) and \( \psi'(w) = 1 \).

In the present paper we characterize the eigenfunctions of a weighted composition operators on \( H(\mathbb{U}) \).

### 2. Main Result

From now on, we assume that \( w \) is the Denjoy-Wolff point of non-elliptic holomorphic self-map \( \psi \) and \( \varphi \) is a holomorphic function on \( \mathbb{U} \) which is continuous at \( w \) and \( \varphi(w) \neq 0 \).

We characterize the eigenfunctions of \( C_{\varphi, \psi} \) in \( H(\mathbb{U}) \). In fact, if the infinite product \( \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z)) \) converges uniformly on compact subsets \( \mathbb{U} \) then, the function

\[
g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z))
\]  

is holomorphic on \( \mathbb{U} \) and satisfies the equation \( \varphi \cdot g \circ \psi = \varphi(w)g \) and is indeed an eigenfunction of \( C_{\varphi, \psi} \). That all eigenfunctions of \( C_{\varphi, \psi} \) in
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$H(U)$, continuous at $w$, are obtained in this way is the content of the following theorem.

**Proposition 2.1.** Let $g \in H(U)$ be a non-zero eigenfunction of $C_{\varphi, \psi}$ which is continuous at $w$. Then either $g(w) = 0$ or the infinite product $\prod_{n=0}^{\infty} \frac{1}{\varphi^n(w)} \varphi^n(\psi(z))$ converges uniformly on compact subsets of $U$ and

$$g(z) = g(w) \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi^n(\psi(z)).$$

(2)

**Proof.** Let $g(w) \neq 0$ and $\varphi \circ g \circ \psi = \lambda g$ for some non-zero scalar $\lambda$. Then $\lambda = \varphi(w)$ and for integer $n \geq 1$,

$$\left( \prod_{i=0}^{n-1} \varphi(\psi_i(z)) \right) g(\psi_n(z)) = \lambda^n g(z) = \varphi(w)^n g(z)$$

and so

$$\left( \prod_{i=0}^{n-1} \frac{1}{\varphi(\psi_i(z))} \right) g(\psi_n(z)) = g(z) \quad (z \in U, n \geq 1) \quad (3)$$

where $\psi_0$ is the identity map on $U$. Since $g(\psi_n(z)) \to g(w)$, the infinite product $\prod_{i=0}^{+\infty} \frac{1}{\varphi(\psi_i(z))}$ converges in $H(U)$ to $g(w)^{-1} g(z)$ and (2) is deduced. □

The next proposition shows the iterate sequence of holomorphic self maps can exhibit a stronger form of convergence to the Denjoy-Wolff point.

**Proposition 2.2.** The series

$$\sum_{n=1}^{+\infty} |\psi_n(z) - w|^\beta$$

(4)

converges uniformly on compact subsets of $U$ whenever

1. $\psi$ is not parabolic and $\beta > 0$, or
2. $\psi$ is parabolic automorphic and $\beta = 2$. 
Proof. Suppose \( \psi \) is not of parabolic type. Then it is either of dilation or hyperbolic type. Let \( \psi \) be of dilation type and \( w \in U \). Then zero is the Denjoy-Wolff point of the self map \( \alpha_w \circ \psi \circ \alpha_w \) where \( \alpha_w(z) = \frac{z-w}{1-wz} \). Choose \( \delta > 0 \) with \( |\psi'(w)| < \delta < 1 \). So \( |\alpha_w \circ \psi \circ \alpha_w(z)| < \delta |z| \) when \( z \) is sufficiently near to zero. If \( K \) is a compact subset of \( U \), then by the Denjoy-Wolff Theorem, \( \alpha_w \circ \psi_n \circ \alpha_w \rightarrow 0 \) uniformly on \( K \) and \( |\alpha_w \circ \psi_{n+k} \circ \alpha_w(z)| < \delta^k |\alpha_w \circ \psi_n \circ \alpha_w(z)| \) for sufficiently large \( n \), every positive integer \( k \), and \( z \in K \). Upon replacing \( \alpha_w(z) \) instead of \( z \) in the previous inequality, we get

\[
\frac{|\psi_{n+k}(z) - w|}{2} \leq |\alpha_w(\psi_{n+k}(z))| \leq \delta^k |\alpha_w(\psi_n(z))| \tag{5}
\]

Now suppose \( \psi \) is hyperbolic and \( w \in \partial U \), then \( 0 < \psi'(w) < 1 \) and by Julia-Caratheodory Inequality ([1], Theorem 3.1) we get

\[
\frac{|\psi(z) - w|^2}{1 - |\psi(z)|^2} < \psi'(w) \frac{|z - w|^2}{1 - |z|^2} \quad (z \in U).
\]

By substituting \( \psi_n(z) \) for \( z \), we get

\[
\frac{|\psi_n(z) - w|^2}{1 - |\psi_n(z)|^2} < (\psi'(w))^n \frac{|z - w|^2}{1 - |z|^2} \quad (z \in U, n \geq 0)
\]

Now if \( K \) is a compact subset of \( U \), then the right hand of the above inequality is bounded on \( K \). Hence it follows that

\[
|\psi_n(z) - w| < \text{const.}(\psi'(w))^n \quad (z \in K) \tag{6}
\]

Thus the inequality (5) and (6) imply that (4) converges uniformly on compact subsets of \( U \) for \( \beta > 0 \). For the next part let \( \psi \) be of parabolic automorphic type. The Linear-Fractional Model Theorem [2, 8, 9] then provides a function \( \sigma \) holomorphic on \( U \) with values in the right half-plane such that \( \sigma \circ \psi = \sigma + ib \) for some real \( b \neq 0 \). Hence more generally \( \sigma \circ \psi_n = \sigma + nib \). Let \( K \) be an arbitrary compact subset of \( U \). For \( n \geq 1 \), pick \( z_n \in K \) such that \( |\psi_n(z_n)| \leq |\psi_n(z)| \) for all \( z \in K \).

The Blaschke condition for a sequence \( (z_n) \) in \( U \) is equivalent, via the map \( w = \frac{1+z}{1-z} \), to the condition

\[
\sum_n \frac{\text{Rew}_{n}}{|1 + w_n|^2} < \infty \tag{7}
\]
for sequences \((w_n)\) in the right half-plane. Since the sequence \((\sigma(z_n))\) is bounded, so (7) is to be satisfied by the sequence \(w_n = \sigma(z_n) + nib\), which is therefore, the zero-sequence of a bounded holomorphic function \(F\) on the right half-plane (see [3] Theorem 11.3, page 191). The function \(f = F \circ \sigma\) is then a non-constant bounded holomorphic function on \(U\), and for \(n \geq 1:\)

\[
f(\psi_n(z_n)) = F(\sigma(\psi_n(z_n))) = F(\sigma(z_n) + nb) = 0
\]

Thus some nonconstant bounded holomorphic functions on \(U\) vanishes at each point of the sequence \((\psi_n(z_n))\), so that sequence satisfies the Blaschke condition. On the other hand, by the Julia-Caratheodory Inequality,

\[
|\psi_n(z) - w|^2 \leq \text{const}(1 - |\psi_n(z)|^2) \leq \text{const}(1 - |\psi_n(z_n)|^2)
\]
on \(K\). Thus (4) uniformly converges on \(K\) for \(\beta = 2\). \(\square\)

Recall that for any \(w \in U\) and positive real number \(\beta\), we denote by \(\text{Lip}_\beta(w)\), the class of holomorphic functions \(\varphi\) satisfying

\[
|\varphi(z) - \varphi(w)| = O(|z - w|^{\beta}) \quad (z \to w)
\]

(8)

For example if \(\varphi \in H(U)\) is analytic at \(w\), then \(\varphi \in \text{Lip}_\beta(w)\) for \(\beta \in (0, 1]\). Moreover, if \(\varphi^{(i)}(w)\) exists and equal to zero for \(i = 1, \ldots, n\) then \(\varphi \in \text{Lip}_\beta(w)\) for \(\beta \in (0, n + 1]\).

**Theorem 2.3.** Let \(\varphi \in \text{Lip}_\beta(w)\) and \(\varphi(w) \neq 0\) then the function \(g(z)\) defined by equation (1) is an eigenfunction for \(C_{\varphi, \psi}\), whenever

1. \(\psi\) is of dilation type, or
2. \(\psi\) is of hyperbolic type and \(\beta > 0\), or
3. \(\psi\) is of parabolic automorphism type and \(\beta = 2\).

**Proof.** Assume that \(\varphi \in \text{Lip}_\beta(w)\) for some real number \(\beta\) and \(K\) is a compact subset of \(U\). Since \(\psi_n \to w\) uniformly on \(K\), by substituting \(\psi_n(z)\) instead of \(z\) in (8) we get

\[
|\varphi(w) - \varphi(\psi_n(z))| = O(|w - \psi_n(z)|^\beta) \quad (z \in K, n \to \infty)
\]
whence

\[ |1 - \frac{1}{\varphi(w)}\psi_n(z)| = O\left(\frac{1}{|\varphi(w)|}|w - \psi_n(z)|^\beta \right) \quad (z \in K, n \to \infty). \]

Now if \( \psi \) is hyperbolic and \( \beta > 0 \) or \( \psi \) is parabolic automorphism and \( \beta = 2 \) then by pervious Proposition, \( \sum_{n=0}^{\infty} |1 - \frac{1}{\varphi(w)}\psi_n(z)| \) and consequently \( g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z)) \) converges uniformly on \( K \). Thus (1) is indeed an eigenfunction for \( C_{\varphi,\psi} \) and the proof is complete. □

References


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