Eigenfunctions of the Weighted Composition Operators

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Abstract. In the present paper, we characterize the eigenfunctions of a weighted composition operator on space of holomorphic function on the unit disk.

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1. Introduction

A weighted composition operator $C_{\varphi, \psi}$ is an operator that maps $f \in H(U)$, the space of holomorphic functions on the unit disk $U$, into $C_{\varphi, \psi}(f)(z) = \varphi(z)f(\psi(z))$, where $\varphi$ and $\psi$ are analytic functions defined in $U$ such that $\psi(U) \subseteq U$. When $\varphi \equiv 1$, we just have the composition operator $C_{\psi}$ defined by $C_{\psi}(f) = f \circ \psi$.

The eigenfunctions of a composition operator on the classical Hardy space $H^2$, induced by a hyperbolic disk automorphism, are considered in [2, 4, 5] where it has been shown that many eigenfunctions of a composition operator can be found in the doubly cyclic subspace generated by special functions in $H^2$.

Studying the eigenfunctions of weighted composition operators entails a study of the iterate behavior of holomorphic self maps. The holomorphic self maps of $U$ are divided into classes of elliptic and non-elliptic type. The elliptic type is an automorphism and has a fixed point in $U$. It is

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well known that this map is conjugate to a rotation $z \rightarrow \lambda z$ for some complex number $\lambda$ with $|\lambda| = 1$. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem [1, 6, 7]. In the following notation $\lim_{k \to \infty}$ means uniformly converges on compact subsets of $\mathbb{U}$ and $\psi_n$ denotes the composition of $\psi$ with itself $n$-times.

**Denjoy-Wolff Iteration Theorem.** Suppose $\psi$ is a holomorphic self-map of $\mathbb{U}$ that is not an elliptic automorphism. Then

(i) If $\psi$ has a fixed point $w \in \mathbb{U}$, then $\psi_n \lim_{k \to \infty} w$ and $|\psi'(w)| < 1$.

(ii) If $\psi$ has no fixed point in $\mathbb{U}$, then there is a point $w \in \partial \mathbb{U}$ such that $\psi_n \lim_{k \to \infty} w$ and the angular derivative of $\psi$ exists at $w$, with $0 < \psi'(w) \leq 1$. We call the unique attracting point $w$, the Denjoy-Wolff point of $\psi$. By the Denjoy-Wolff Iteration Theorem, a general classification of a non-elliptic holomorphic self maps of $\mathbb{U}$ can be given: let $w$ be the Denjoy-Wolff point of a holomorphic self-map of $\mathbb{U}$. We say $\psi$ is of dilation type if $w \in \mathbb{U}$, of hyperbolic type if $w \in \partial \mathbb{U}$ and $\psi'(w) < 1$, and of parabolic type if $w \in \partial \mathbb{U}$ and $\psi'(w) = 1$.

In the present paper we characterize the eigenfunctions of a weighted composition operators on $H(\mathbb{U})$.

### 2. Main Result

From now on, we assume that $w$ is the Denjoy-Wolff point of non-elliptic holomorphic self-map $\psi$ and $\varphi$ is a holomorphic function on $\mathbb{U}$ which is continuous at $w$ and $\varphi(w) \neq 0$.

We characterize the eigenfunctions of $C_{\varphi, \psi}$ in $H(\mathbb{U})$. In fact, if the infinite product $\prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z))$ converges uniformly on compact subsets $\mathbb{U}$ then, the function

$$ g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z)) $$

(1)

is holomorphic on $\mathbb{U}$ and satisfies the equation $\varphi.g \circ \psi = \varphi(w)g$ and is indeed an eigenfunction of $C_{\varphi, \psi}$. That all eigenfunctions of $C_{\varphi, \psi}$ in
$H(\mathbb{U})$, continuous at $w$, are obtained in this way is the content of the following theorem.

**Proposition 2.1.** Let $g \in H(\mathbb{U})$ be a non-zero eigenfunction of $C_{\varphi, \psi}$ which is continuous at $w$. Then either $g(w) = 0$ or the infinite product $\prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z))$ converges uniformly on compact subsets of $\mathbb{U}$ and

$$g(z) = g(w) \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z)).$$  \hspace{1cm} (2)

**Proof.** Let $g(w) \neq 0$ and $\varphi \circ \psi = \lambda g$ for some non-zero scalar $\lambda$. Then $\lambda = \varphi(w)$ and for integer $n \geq 1$,

$$\left( \prod_{i=0}^{n-1} \varphi(\psi_i(z)) \right) g(\psi_n(z)) = \lambda^n g(z) = \varphi(w)^n g(z)$$

and so

$$\left( \prod_{i=0}^{n-1} \frac{1}{\varphi(w)} \varphi(\psi_i(z)) \right) g(\psi_n(z)) = g(z) \quad (z \in \mathbb{U}, n \geq 1)$$ \hspace{1cm} (3)

where $\psi_0$ is the identity map on $\mathbb{U}$. Since $g(\psi_n(z)) \rightarrow g(w)$, the infinite product $\prod_{i=0}^{+\infty} \frac{1}{\varphi(w)} \varphi(\psi_i(z))$ converges in $H(\mathbb{U})$ to $g(w)^{-1} g(z)$ and (2) is deduced. \hfill \Box

The next proposition shows the iterate sequence of holomorphic self maps can exhibit a stronger form of convergence to the Denjoy-Wolff point.

**Proposition 2.2.** The series

$$\sum_{n=1}^{+\infty} |\psi_n(z) - w|^\beta$$ \hspace{1cm} (4)

converges uniformly on compact subsets of $\mathbb{U}$ whenever

1. $\psi$ is not parabolic and $\beta > 0$, or
2. $\psi$ is parabolic automorphic and $\beta = 2$. 


Proof. Suppose $\psi$ is not of parabolic type. Then it is either of dilation or hyperbolic type. Let $\psi$ be of dilation type and $w \in \mathbb{U}$. Then zero is the Denjoy-Wolff point of the self map $\alpha_w \circ \psi \circ \alpha_w$ where $\alpha_w(z) = \frac{z-w}{1-wz}$.

Choose $\delta > 0$ with $|\psi'(w)| < \delta < 1$. So $|\alpha_w \circ \psi \circ \alpha_w(z)| < \delta |z|$ when $z$ is sufficiently near to zero. If $K$ is a compact subset of $\mathbb{U}$, then by the Denjoy-Wolff Theorem, $\alpha_w \circ \psi_n \circ \alpha_w \rightarrow 0$ uniformly on $K$ and $|\alpha_w \circ \psi_{n+k} \circ \alpha_w(z)| < \delta^k |\alpha_w \circ \psi_n \circ \alpha_w(z)|$ for sufficiently large $n$, every positive integer $k$, and $z \in K$. Upon replacing $\alpha_w(z)$ instead of $z$ in the previous inequality, we get

$$\frac{|\psi_{n+k}(z) - w|}{2} \leq |\alpha_w(\psi_{n+k}(z))| \leq \delta^k |\alpha_w(\psi_n(z))|$$ (5)

Now suppose $\psi$ is hyperbolic and $w \in \partial \mathbb{U}$, then $0 < \psi'(w) < 1$ and by Julia-Caratheodory Inequality ([1], Theorem 3.1) we get

$$\frac{|\psi(z) - w|^2}{1 - |\psi(z)|^2} < \psi'(w) \frac{|z - w|^2}{1 - |z|^2} \quad (z \in \mathbb{U}).$$

By substituting $\psi_n(z)$ for $z$, we get

$$\frac{|\psi_n(z) - w|^2}{1 - |\psi_n(z)|^2} < (\psi'(w))^n \frac{|z - w|^2}{1 - |z|^2} \quad (z \in \mathbb{U}, n \geq 0)$$

Now if $K$ is a compact subset of $\mathbb{U}$, then the right hand of the above inequality is bounded on $K$. Hence it follows that

$$|\psi_n(z) - w| < \text{const.}(\psi'(w))^\frac{z}{2} \quad (z \in K)$$ (6)

Thus the inequality (5) and (6) imply that (4) converges uniformly on compact subsets of $\mathbb{U}$ for $\beta > 0$. For the next part let $\psi$ be of parabolic automorphic type. The Linear-Fractional Model Theorem [2, 8, 9] then provides a function $\sigma$ holomorphic on $\mathbb{U}$ with values in the right half-plane such that $\sigma \circ \psi = \sigma + ib$ for some real $b \neq 0$. Hence more generally $\sigma \circ \psi_n = \sigma + nib$. Let $K$ be an arbitrary compact subset of $\mathbb{U}$. For $n \geq 1$, pick $z_n \in K$ such that $|\psi_n(z_n)| \leq |\psi_n(z)|$ for all $z \in K$.

The Blaschke condition for a sequence $(z_n)$ in $\mathbb{U}$ is equivalent, via the map $w = \frac{1+z}{1-z}$, to the condition

$$\sum_n \frac{\text{Rew}_n}{|1 + w_n|^2} < \infty \quad (7)$$
for sequences \((w_n)\) in the right half-plane. Since the sequence \((\sigma(z_n))\) is bounded, so (7) is to be satisfied by the sequence \(w_n = \sigma(z_n) + nib\), which is therefore, the zero-sequence of a bounded holomorphic function \(F\) on the right half-plane (see [3] Theorem 11.3, page 191). The function \(f = F \circ \sigma\) is then a non-constant bounded holomorphic function on \(U\), and for \(n \geq 1:\)

\[
f(\psi_n(z_n)) = F(\psi_n(z_n)) = F(\sigma(z_n) + nbi) = 0
\]

Thus some nonconstant bounded holomorphic functions on \(U\) vanishes at each point of the sequence \((\psi_n(z_n))\), so that sequence satisfies the Blaschke condition. On the other hand, by the Julia-Caratheodory Inequality,

\[
|\psi_n(z) - w|^2 \leq \text{const}(1 - |\psi_n(z)|^2) \leq \text{const}(1 - |\psi_n(z_n)|^2)
\]
on \(K\). Thus (4) uniformly converges on \(K\) for \(\beta = 2\). \(\Box\)

Recall that for any \(w \in U\) and positive real number \(\beta\), we denote by \(Lip_\beta(w)\), the class of holomorphic functions \(\varphi\) satisfying

\[
|\varphi(z) - \varphi(w)| = O(|z - w|^{\beta}) \quad (z \to w)
\]

For example if \(\varphi \in H(U)\) is analytic at \(w\), then \(\varphi \in Lip_\beta(w)\) for \(\beta \in (0, 1]\). Moreover, if \(\varphi^{(i)}(w)\) exists and equal to zero for \(i = 1, \ldots, n\) then \(\varphi \in Lip_\beta(w)\) for \(\beta \in (0, n + 1]\).

**Theorem 2.3.** Let \(\varphi \in Lip_\beta(w)\) and \(\varphi(w) \neq 0\) then the function \(g(z)\) defined by equation (1) is an eigenfunction for \(C_{\varphi, \psi}\), whenever

1. \(\psi\) is of dilation type, or
2. \(\psi\) is of hyperbolic type and \(\beta > 0\), or
3. \(\psi\) is of parabolic automorphism type and \(\beta = 2\).

**Proof.** Assume that \(\varphi \in Lip_\beta(w)\) for some real number \(\beta\) and \(K\) is a compact subset of \(U\). Since \(\psi_n \to w\) uniformly on \(K\), by substituting \(\psi_n(z)\) instead of \(z\) in (8) we get

\[
|\varphi(w) - \varphi(\psi_n(z))| = O(|w - \psi_n(z)|^\beta) \quad (z \in K, n \to \infty)
\]
whence

\[ |1 - \frac{1}{\varphi(w)}\varphi_n(z)| = O\left(\frac{1}{|\varphi(w)|}|w - \psi_n(z)|^\beta \right) \quad (z \in K, n \to \infty). \]

Now if \( \psi \) is hyperbolic and \( \beta > 0 \) or \( \psi \) is parabolic automorphism and \( \beta = 2 \) then by previous Proposition, \( \sum_{n=0}^{\infty} |1 - \frac{1}{\varphi(w)}\varphi(\psi_n(z))| \) and consequently \( g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z)) \) converges uniformly on \( K \). Thus (1) is indeed an eigenfunction for \( C_{\varphi,\psi} \) and the proof is complete. \( \square \)

References


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