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Semismooth Function on Riemannian Manifolds

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Abstract. In this paper, We extend the concept of semismoothness for functions to the Riemannian manifolds setting. Then, some properties of these functions are studied.

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1. Introduction

Semismoothness was originally introduced by Mifflin (see[3]) for functional. For function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the concept of semismoothness is equivalent to the uniform convergence of directional derivatives in all directions.
Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be locally Lipschitz and D_F denote the set where F is differentiable. Then the Clark generalized Jacobian of F at x denoted by $\partial_{cl}F(x)$ is defined as

$$\partial_{cl}F(x) := co\{\lim_{x_n \longrightarrow x} JF(x_n) \mid x_n \longrightarrow x, x_n \in D_F\},$$

where "J" denotes Jacobian and "co" stands for convex hull.

Definition 1.1. *We say that a locally Lipschitz function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is semismooth at x if*

$$\lim_{v \in \partial_{cl}F(x+th'),\ h' \longrightarrow h,\ t \downarrow 0^+} vh', \tag{1}$$

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exists for any $h \in \mathbb{R}^n$*.*
Convex functions, smooth functions and maximums of smooth functions are semismooth. Smooth compositions of semismooth functions are still semismooth. It was shown that a function F from $\mathbb{R}^n$ to $\mathbb{R}^m$ is semismooth if and only if all its components are semismooth. The proof of the following theorems can be found in [6].

Theorem 1.2. *If* F *is semismooth, then the directional derivative*

$$F'(x;h) = \lim_{t \longrightarrow 0^+} \frac{1}{t}[F(x+th) - F(x)],$$

for $h \in \mathbb{R}^n$ *exists and is equal to (2.1), i.e,*

$$F'(x;h) = \lim_{v \in \partial_{cl} F(x+th'),\ h' \longrightarrow h,\ t \downarrow 0^+} vh',$$

for $h \in \mathbb{R}^n$.

Lemma 1.3. *Suppose that* $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is a locally Lipschitz function and* $F'(x;h)$ *exists for each* h *at* x*. Then*

(i) $F'(x;.)$ *is Lipschitz.*
(ii) for each h*, there exists a* $v \in \partial_{cl} F(x)$ *such that*

$$F'(x;h) = vh.$$

In the following, we introduce some fundamental properties and notations of Riemannian manifolds.

Definition 1.4. *A real-valued function* f *defined on a complete Riemannian manifold* M *is said to be a convex if* f *is convex when restricted to any geodesics of* M*, which means that*

$$(fo\gamma)(ta + (1-t)b) \leqslant tf(\gamma(a)) + (1-t)f(\gamma(b)),$$

holds for any $a, b \in \mathbb{R}$ *and* $0 \leqslant t \leqslant 1$.

Definition 1.5. *A real-valued function f defined on a complete Riemannian manifold M is said to be Lipschitz if there exists a constant $L(M) = L \geqslant 0$ such that*

$$| f(p) - f(q) | \leqslant Ld(p, q), \tag{2}$$

for all $p, p' \in M$, where d is the Riemannian distance on M.
Besides this global concept, if for each $p_0 \in M$, there exists $L(p_0) \geqslant 0$ and $\delta = \delta(p_0) > 0$ such that Inequality (2.2) occurs with $L = L(p_0)$, for all $p, q \in B_\delta(p_0) := \{p \in M \mid d(p_0, p) < \delta\}$, then f is called locally Lipschitz.

Definition 1.6. *Let M be a complete Riemannian manifold and let $f : M \longrightarrow \mathbb{R}$ be a convex function. Then the directional derivative of f at p in the direction $v \in T_pM$ is defined by*

$$f'(p, v) = \lim_{t \longrightarrow 0^+} q_{\gamma_v}(t) = \inf_{t>0} q_{\gamma_v}(t),$$

where $\gamma_v : \mathbb{R} \longrightarrow M$ is the geodesic such that $\gamma_v(0) = p, \gamma_v'(0) = v$ and

$$q_\gamma(t) = \frac{f(\gamma(t)) - f(p)}{t}.$$

Definition 1.7. *Let $f : M \longrightarrow \mathbb{R}$ be a locally Lipschitz function and (U, φ) be a chart around $p \in M$. Then the clarke generalized Jacobian of f at p in the direction of $v \in T_pM$ is defined by*

$$f^0(p, v) = \limsup_{t \downarrow 0, y \longrightarrow x} \frac{f o \varphi^{-1}(y + tv) - f o \varphi^{-1}(y)}{t},$$

where $\varphi(p) = x$. (see [2])

2. Semismoothness on Riemannian Manifolds

Definition 2.1. *We say that a locally Lipschitz function $f : M \longrightarrow \mathbb{R}$ is semismooth at p, If there exists a chart (U, φ) at p such that $fo\varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is semismooth at $\varphi(p) = x \in \mathbb{R}^n$. It means that*

$$\lim_{v\in\partial_{cl}(fo\varphi^{-1})(x+th),t\downarrow 0^+} vh, \tag{3}$$

exists for all $h \in \mathbb{R}^n$.

Note that by using a normal chart (U, φ) at p, the formula (3) gives us

$$\lim_{v\in\partial_{cl}f(\exp_p th),t\downarrow 0^+} vh, \tag{4}$$

for all $h \in T_pM \cong \mathbb{R}^n$.

In particular, observe that if $M = \mathbb{R}^n$, (4) implies (3).

Proposition 2.2. *The above definition does not depend on the coordinate system.*

Proof. Suppose that f is semismooth at p i.e. there exists a chart (U, φ) at p such that $fo\varphi^{-1}$ is semismooth at $\varphi(p)$. Now if there exists another chart such as (v, ψ) at p, we shall show that f in this chart is also semismooth, i.e. $fo\psi^{-1}$ at $\psi(p)$ is semismooth.
We consider

$$fo\psi^{-1} = fo\varphi^{-1}o\varphi o\psi^{-1},$$

by assumption $fo\varphi^{-1}$ is semismooth and according to the C^∞ structure on M, the combination $\varphi o\psi^{-1}$ is smooth and according to the properties of the resulting semismooth functions (see [4]), this combination is semismooth. Hence f is also semismooth in this chart and therefore the concept of semismoothness on manifolds does not depend on the coordinate system. $\square$

Theorem 2.3. *Suppose that $f : M \longrightarrow \mathbb{R}$ is semismooth, then the directional derivative*

$$f'(p;h) = \lim_{t \longrightarrow 0^+} \frac{f(\gamma(t)) - f(p)}{t},$$

exists and is equal to

$$f'(p;h) = \lim_{v \in \partial_{cl} f(\exp_p th), t \downarrow 0} vh.$$

where $\gamma : \mathbb{R} \longrightarrow M$ is geodesic and $\gamma(0) = p, \gamma'(0) = h$.

Proof. Since $f : M \longrightarrow \mathbb{R}$ is semismooth, there is chart (U, φ) at p such that $fo\varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ at $\varphi(p) = x \in \mathbb{R}^n$ is semismooth, As a result of theorem (2.2), the directional derivative

$$(fo\varphi^{-1})'(x;h) = \lim_{t \longrightarrow 0} \frac{1}{t}[fo\varphi^{-1}(x+th) - fo\varphi^{-1}(x)],$$

exists and is equal to

$$(fo\varphi^{-1})'(x;h) = \lim_{v \in \partial_{cl}(fo\varphi^{-1})(x+th), t \downarrow 0} vh. \tag{5}$$

Left side of the above equality with respect to the normal coordinate system and property of exponential function ($\gamma_h(t) = \exp(th)$), can be written as follows

$$\lim_{t \longrightarrow 0^+} \frac{fo\varphi^{-1}(x+th) - fo\varphi^{-1}(x)}{t} = \lim_{t \longrightarrow 0^+} \frac{f(\gamma_h(t)) - f(p)}{t} = f'(p;h), \tag{6}$$

and also consider the right side of (5) as follows

$$\lim_{v \in \partial_{cl}(fo\varphi^{-1})(x+th), t \downarrow 0} vh = \lim_{v \in \partial_{cl}(f(\gamma_h(t)), t \downarrow 0} vh. \tag{7}$$

As a result of the (5), (6) and (7), one has that

$$f'(p;h) = \lim_{v \in \partial_{cl} f(\exp_p th), t \downarrow 0} vh.$$

This completes the proof. □

Theorem 2.4. *Suppose that $f : M \longrightarrow \mathbb{R}$ is convex in the neighborhood of $p \in M$. Then f is semismooth at p.*

Proof. For every sequence $\{p_k\}$ converges to $p(p_k \neq p)$ and for every sequence $\{v_k\}$, $v_k \in \partial_{cl} f(p_k)$, we have

$$\lim_{k \longrightarrow \infty} f'(p; d_k) = \lim_{k \longrightarrow \infty} (v_k)^T d_k, \tag{8}$$

where

$$d_k \equiv \frac{\exp_{p_k}^{-1} p}{\| \exp_{p_k}^{-1} p \|_{p_k}}.$$

Without loss of generality, can assume

$$\lim_{k \longrightarrow \infty} d_k = d, \qquad \lim_{k \longrightarrow \infty} v_k = v \in \partial_{cl} f(p).$$

Since the left and the right limit of (8) are equal respectively with $f'(p; d)$ and $v^T d$, Then

$$v^T d = f'(p; d).$$

Since f is convex and $v_k \in \partial_{cl} f(p_k)$, we have

$$f(p) - f(p_k) \geqslant < v_k, \exp_{p_k}^{-1} p >,$$

and consider $k \longrightarrow \infty$,

$$v^T d \geqslant f'(p; d),$$

and since $v \in \partial_{cl} f(p)$, therefore, we have

$$f'(p; d) \geqslant v^T d.$$

Thus equality is established and the proof is completed. □

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