Semismooth Function on Riemannian Manifolds

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Abstract. In this paper, we extend the concept of semismoothness for functions to the Riemannian manifolds setting. Then, some properties of these functions are studied.

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1. Introduction

Semismoothness was originally introduced by Mifflin (see[3]) for functional. For function \( F : \mathbb{R}^n \to \mathbb{R}^n \), the concept of semismoothness is equivalent to the uniform convergence of directional derivatives in all directions.

Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be locally Lipschitz and \( D_F \) denote the set where \( F \) is differentiable. Then the Clark generalized Jacobian of \( F \) at \( x \) denoted by \( \partial_{cl} F(x) \) is defined as

\[
\partial_{cl} F(x) := \text{co} \{ \lim_{x_n \to x} JF(x_n) \mid x_n \to x, x_n \in D_F \},
\]

where "J" denotes Jacobian and "co" stands for convex hull.

**Definition 1.1.** We say that a locally Lipschitz function \( F : \mathbb{R}^n \to \mathbb{R}^m \) is semismooth at \( x \) if

\[
\lim_{v \in \partial_{cl} F(x+th'), h' \to h, t \to 0^+} vh',
\]

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exists for any $h \in \mathbb{R}^n$.

Convex functions, smooth functions and maximums of smooth functions are semismooth. Smooth compositions of semismooth functions are still semismooth. It was shown that a function $F$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is semismooth if and only if all its components are semismooth. The proof of the following theorems can be found in [6].

**Theorem 1.2.** If $F$ is semismooth, then the directional derivative

$$F'(x; h) = \lim_{t \to 0^+} \frac{1}{t}[F(x + th) - F(x)],$$

for $h \in \mathbb{R}^n$ exists and is equal to (2.1), i.e,

$$F'(x; h) = \lim_{v \in \partial_{cl} F(x + th'), h' \to h, t \to 0^+} vh',$$

for $h \in \mathbb{R}^n$.

**Lemma 1.3.** Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz function and $F'(x; h)$ exists for each $h$ at $x$. Then

(i) $F'(x; .)$ is Lipschitz.
(ii) for each $h$, there exists a $v \in \partial_{cl} F(x)$ such that

$$F'(x; h) = vh.$$

In the following, we introduce some fundamental properties and notations of Riemannian manifolds.

**Definition 1.4.** A real-valued function $f$ defined on a complete Riemannian manifold $M$ is said to be a convex if $f$ is convex when restricted to any geodesics of $M$, which means that

$$(f \circ \gamma)(ta + (1 - t)b) \leq tf(\gamma(a)) + (1 - t)f(\gamma(b)),$$
holds for any $a, b \in \mathbb{R}$ and $0 \leq t \leq 1$.

**Definition 1.5.** A real-valued function $f$ defined on a complete Riemannian manifold $M$ is said to be Lipschitz if there exists a constant $L(M) = L \geq 0$ such that

$$|f(p) - f(q)| \leq Ld(p, q),$$

(2)

for all $p, p' \in M$, where $d$ is the Riemannian distance on $M$.

Besides this global concept, if for each $p_0 \in M$, there exists $L(p_0) \geq 0$ and $\delta = \delta(p_0) > 0$ such that Inequality (2.2) occurs with $L = L(p_0)$, for all $p, q \in B_\delta(p_0) := \{p \in M \mid d(p_0, p) < \delta\}$, then $f$ is called locally Lipschitz.

**Definition 1.6.** Let $M$ be a complete Riemannian manifold and let $f : M \rightarrow \mathbb{R}$ be a convex function. Then the directional derivative of $f$ at $p$ in the direction $v \in T_pM$ is defined by

$$f'(p, v) = \lim_{t \to 0^+} q_{p_\gamma_v}(t) = \inf_{t > 0} q_{p_\gamma_v}(t),$$

where $\gamma_v : \mathbb{R} \rightarrow M$ is the geodesic such that $\gamma_v(0) = p, \gamma_v'(0) = v$ and

$$q_{p_\gamma}(t) = \frac{f(\gamma(t)) - f(p)}{t}.$$

**Definition 1.7.** Let $f : M \rightarrow \mathbb{R}$ be a locally Lipschitz function and $(U, \varphi)$ be a chart around $p \in M$. Then the clark generalized Jacobian of $f$ at $p$ in the direction of $v \in T_pM$ is defined by

$$f^0(p, v) = \lim_{t \to 0^+} \sup_{s \to x} \frac{f \circ \varphi^{-1}(y + tv) - f \circ \varphi^{-1}(y)}{t},$$

where $\varphi(p) = x$. (see [2])
2. Semismoothness on Riemannian Manifolds

**Definition 2.1.** We say that a locally Lipschitz function $f : M \rightarrow \mathbb{R}$ is semismooth at $p$, if there exists a chart $(U, \varphi)$ at $p$ such that $f \circ \varphi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ is semismooth at $\varphi(p) = x \in \mathbb{R}^n$. It means that

$$\lim_{v \in \partial_+(f \circ \varphi^{-1})(x + th), t \uparrow 0^+} vh,$$

exists for all $h \in \mathbb{R}^n$.

Note that by using a normal chart $(U, \varphi)$ at $p$, the formula (3) gives us

$$\lim_{v \in \partial_+(f \circ \psi^{-1})(\exp_p, th), t \uparrow 0^+} vh,$$

for all $h \in T_pM \cong \mathbb{R}^n$.

In particular, observe that if $M = \mathbb{R}^n$, (4) implies (3).

**Proposition 2.2.** The above definition does not depend on the coordinate system.

**Proof.** Suppose that $f$ is semismooth at $p$ i.e. there exists a chart $(U, \varphi)$ at $p$ such that $f \circ \varphi^{-1}$ is semismooth at $\varphi(p)$. Now if there exists another chart such as $(V, \psi)$ at $p$, we shall show that $f$ in this chart is also semismooth, i.e. $f \circ \psi^{-1}$ at $\psi(p)$ is semismooth.

We consider

$$f \circ \psi^{-1} = f \circ \varphi^{-1} \circ \varphi \circ \psi^{-1},$$

by assumption $f \circ \varphi^{-1}$ is semismooth and according to the $C^\infty$ structure on $M$, the combination $\varphi \circ \psi^{-1}$ is smooth and according to the properties of the resulting semismooth functions (see [4]), this combination is semismooth. Hence $f$ is also semismooth in this chart and therefore the concept of semismoothness on manifolds does not depend on the coordinate system. $\square$
Theorem 2.3. Suppose that \( f : M \to \mathbb{R} \) is semismooth, then the directional derivative

\[
f'(p; h) = \lim_{t \to 0^+} \frac{f(\gamma(t)) - f(p)}{t},
\]

exists and is equal to

\[
f'(p; h) = \lim_{v \in \partial_{cl} f(p \cdot th), t \downarrow 0} vh.
\]

where \( \gamma : \mathbb{R} \to M \) is geodesic and \( \gamma(0) = p, \gamma'(0) = h. \)

Proof. Since \( f : M \to \mathbb{R} \) is semismooth, there is chart \( (U, \varphi) \) at \( p \) such that \( f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R} \) at \( \varphi(p) = x \in \mathbb{R}^n \) is semismooth, As a result of theorem (2.2), the directional derivative

\[
(f \circ \varphi^{-1})'(x; h) = \lim_{t \to 0^+} \frac{1}{t} [f \circ \varphi^{-1}(x + th) - f \circ \varphi^{-1}(x)],
\]

exists and is equal to

\[
(f \circ \varphi^{-1})'(x; h) = \lim_{v \in \partial_{cl} (f \circ \varphi^{-1})(x + th), t \downarrow 0} vh. \tag{5}
\]

Left side of the above equality with respect to the normal coordinate system and property of exponential function \( (\gamma_h(t) = \exp(th)) \), can be written as follows

\[
\lim_{t \to 0^+} \frac{f \circ \varphi^{-1}(x + th) - f \circ \varphi^{-1}(x)}{t} = \lim_{t \to 0^+} \frac{f(\gamma_h(t)) - f(p)}{t} = f'(p; h), \tag{6}
\]

and also consider the right side of (5) as follows

\[
\lim_{v \in \partial_{cl} (f \circ \varphi^{-1})(x + th), t \downarrow 0} vh = \lim_{v \in \partial_{cl} (f(\gamma_h(t)), t \downarrow 0} vh. \tag{7}
\]

As a result of the (5), (6) and (7), one has that

\[
f'(p; h) = \lim_{v \in \partial_{cl} f(p \cdot th), t \downarrow 0} vh.
\]
This completes the proof. □

**Theorem 2.4.** Suppose that \( f : M \to \mathbb{R} \) is convex in the neighborhood of \( p \in M \). Then \( f \) is semismooth at \( p \).

**Proof.** For every sequence \( \{p_k\} \) converges to \( p(p_k \neq p) \) and for every sequence \( \{v_k\}, v_k \in \partial_{cl} f(p_k) \), we have

\[
\lim_{k \to \infty} f'(p; d_k) = \lim_{k \to \infty} (v_k)^T d_k, \tag{8}
\]

where

\[
d_k = \frac{\exp_{p_k}^{-1} p}{\|\exp_{p_k}^{-1} p\|_{p_k}}.
\]

Without loss of generality, can assume

\[
\lim_{k \to \infty} d_k = d, \quad \lim_{k \to \infty} v_k = v \in \partial_{cl} f(p).
\]

Since the left and the right limit of (8) are equal respectively with \( f'(p; d) \) and \( v^T d \), Then

\[
v^T d = f'(p; d).
\]

Since \( f \) is convex and \( v_k \in \partial_{cl} f(p_k) \), we have

\[
f(p) - f(p_k) \geq < v_k, \exp_{p_k}^{-1} p >,
\]

and consider \( k \to \infty \)

\[
v^T d \geq f'(p; d),
\]

and since \( v \in \partial_{cl} f(p) \), therefore, we have

\[
f'(p; d) \geq v^T d.
\]

Thus equality is established and the proof is completed. □
References


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