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New Operators for Fractional Integration Theory with Some Applications

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Abstract. In this paper, we introduce new generalizations for the well konwn (k,s,h)-Riemann-Liouville, (k,s)-Hadamard and (k,s,h)-Hadamard fractional integral operators. We prove some of their properties. Then, using our proposed approaches, we establish some applications on inequalities.

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1. Introduction

In 1993 [17] Samko, Kilbas and Marichev have introduced the fractional integration with respect to another function g it given by:

$$J_{a,g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) \, dt.$$

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Then, in 2011, [11] Katugampola has presented the following generalization:

$$\begin{aligned}&\int_a^x t_1^s dt_1 \int_a^{t_1} t_2^s dt_2 ... \int_a^{t_{n-1}} t_n^s dt_n \\ = &\frac{(s+1)^{1-n}}{\Gamma(n)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{n-1} t^s f(t)\, dt, \ n \in \mathbb{N}^*.\end{aligned}$$

For $\alpha > 0, \ s \in - \{-1\}$, the fractional integral was given by

$$^{s}J_a^{\alpha} f(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\alpha-1} t^s f(t)\, dt.$$

In [14], Mubeen and Habibullah have introduced the following $k-$Riemann-Liouville fractional integral:

$$_{k}J_a^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} t^s f(t)\, dt, \ \alpha > 0, x > a,$$

where $k > 0$ and $\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{u^k}{k}} u^{\alpha-1} du, \ \alpha > 0.$

Very recently, Sarikaya et al. [19] have elaborated another approach for the $(k,s)-$Riemann-Liouville fractional integration. The related definition is given by:

$$^{s}_{k}J_a^{\alpha} f(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s f(t)\, dt.$$

Many researchers have been concerned with the fractional integral theory with its applications. For more details, we refer to [4, 5, 6, 7, 8, 11, 18, 19, 21, 23].

Our purpose in this paper is to present new generalizations for the above cited approaches by introducing new integral operators related to the fractional integration theory. Then, we prove some of their properties of semi group and commutativity properties. Some applications for the introduced operators are also discussed.

2. (k,s,h) Riemann-Liouville, (k,s)-Hadamard and (k,s,h)-Hadamard Integral Operators

In this section, we begin by recalling the fractional integration definitions in the sense of Riemann-Liouville and those of Hadamard. Then, we introduce new concepts that generalize the previous definitions. Some properties of the introduced approaches are also discussed. From the papers [14,17,19], we present:

Definition 2.1. *The Hadamard fractional integral of order* $\alpha \in^{+}$ *of a function* $f(t)$*, for all* $0 < a < t < \infty$*, is defined as*

$$I_a^{\alpha}\left(f\left(t\right)\right) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{f(\tau)}{\tau}d\tau;\ \alpha \geqslant 0,\ 0 < a \leqslant \tau \leqslant t\ , \quad (1)$$

provided the integral exists, where $\Gamma\left(\alpha\right) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$.

Definition 2.2. *The* $k-$*Riemann–Liouville fractional integral of order* $\alpha > 0$*, for a continuous function* f *on* $[a,b]$ *is defined as*

$${}_kJ_a^{\alpha}\left(f\left(t\right)\right) = \frac{1}{k\Gamma_k\left(\alpha\right)}\int_a^t \left(t-\tau\right)^{\frac{\alpha}{k}-1} f\left(\tau\right)d\tau, \quad (2)$$

where $k > 0$, $\Gamma_k\left(\alpha\right) = \int_0^{\infty} e^{-\frac{u^k}{k}} u^{\alpha-1}du, \alpha > 0$.

Definition 2.3. *The* $(k,h)-$*Riemann–Liouville fractional integral of order* $\alpha > 0$*, for a continuous function* f *on* $[a,b]$*, with respect to another measurable, increasing, positive and monotone function* h *on* $(a,b]$ *and* $h\left(t\right)$ *having a continuous derivative* $h'\left(t\right)$ *on* $\left(a,b\right)$*, is defined by*

$${}_kJ_{a,h}^{\alpha}\left(f\left(t\right)\right) = \frac{1}{k\Gamma_k\left(\alpha\right)}\int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1} h'\left(\tau\right)f\left(\tau\right)d\tau. \quad (3)$$

Definition 2.4. *The* $(k,s)-$*Riemann–Liouville fractional integral of order* $\alpha > 0$*, for a continuous function* f *on* $[a,b]$ *is defined as*

$${}_k^sJ_a^{\alpha}\left(f\left(t\right)\right) = \frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_k\left(\alpha\right)}\int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s f\left(\tau\right)d\tau, \quad (4)$$

where $k>0,\ s\in\mathbb{R}\backslash\{-1\}$.

Now, we introduce the $(k,s,h)-$Riemann-Liouville fractional integration as follows:

Definition 2.5. *Let* $f\in L^1[a,b]$ *and* h *be a measurable, increasing, positive, monotone function with* $h\in C^1([a,b])$. *The* $(k,s,h)-$*Riemann–Liouville fractional integral with respect to* h, *is defined by*

$$ {}_k^sJ_{a,h}^{\alpha}\left(f\left(t\right)\right)=\frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_k\left(\alpha\right)}\int_a^t\left(h^{s+1}\left(t\right)-h^{s+1}\left(\tau\right)\right)^{\frac{\alpha}{k}-1}h^s\left(\tau\right)h'\left(\tau\right)f\left(\tau\right)d\tau, \tag{5}$$

where $\alpha>0,\ k>0,\ s\in\mathbb{R}\backslash\{-1\}$.

We introduce also the following definition related to the $(k,h)-$Hadamard integration:

Definition 2.6. *Let* $f\in L^1[a,b]$ *and* h *be a measurable, increasing, positive, monotone function with* $h\in C^1([a,b])$. *The* $(k,h)-$*Hadamard fractional integral with respect to* h *is defined by:*

$$ {}_kI_{a,h}^{\alpha}\left(f\left(t\right)\right)=\tfrac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\log\tfrac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\tfrac{h'(\tau)}{h(\tau)}f\left(\tau\right)d\tau,\ \alpha>0, \tag{6}$$

where $0<a<t\leqslant b, k>0$.

In a more general case, we introduce also the $(k,s,h)-$Hadamard fractional integration as follows:

Definition 2.7. *Let* $f\in L^1[a,b]$ *and* h *be a measurable, increasing, positive, monotone function with* $h\in C^1([a,b])$. *The* $(k,s,h)-$*Hadamard fractional integral with respect to* h *is defined by:*

$$\begin{aligned} {}_k^sI_{a,h}^{\alpha}\left(f\left(t\right)\right) &= \frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_k\left(\alpha\right)}\int_a^t\left(\log^{s+1}h\left(t\right)-\log^{s+1}h\left(\tau\right)\right)^{\frac{\alpha}{k}-1} \\ &\quad\times\log^s h\left(\tau\right)\frac{h'\left(\tau\right)}{h\left(\tau\right)}f\left(\tau\right)d\tau, \end{aligned} \tag{7}$$

where $0<a<t\leqslant b,\ \alpha>0,\ k>0,\ s\in\mathbb{R}\backslash\{-1\}$.

Now, we are able to prove the following properties.

Thanks to Definition 5, we prove:

Theorem 2.8. *The (k,s,h)-Riemann-Liouville integral operator* ${}_{k}^{s}J_{a,h}^{\alpha}$ $f(t)$ *exists for any* $t \in [a,b]$ *and* ${}_{k}^{s}J_{a,h}^{\alpha}$ $f(t) \in L^1[a,b],\ \alpha > 0.$

Proof. Let $T_1 : [a,b] \times [a,b] \to \mathbb{R}$, where

$$\begin{aligned} T_1(t,\tau) &= \left\lfloor \left(h^{s+1}(t) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\, h'(\tau) \right\rfloor_+ \\ &= \begin{cases} \left(h^{s+1}(t) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\, h'(\tau)\,, & a \leqslant \tau < t \leqslant b \\ 0 \qquad , & a \leqslant t < \tau \leqslant b. \end{cases} \end{aligned} \tag{8}$$

Since T_1 is measurable on $[a,b] \times [a,b]$, then we have

$$\begin{aligned} &\left| \int_a^b \left(\int_a^b \left(h^{s+1}(t) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\, h'(\tau)\, f(\tau)\, d\tau \right) dt \right| \\ \leqslant\ & \int_a^b |f(t)| \left(\left| \int_a^t \left(h^{s+1}(t) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\, h'(\tau)\, d\tau \right| \right) dt \\ \leqslant\ & \frac{k}{\alpha\,|s+1|} \int_a^b \left(h^{s+1}(t) - h^{s+1}(a)\right)^{\frac{\alpha}{k}} |f(t)|\, dt \\ \leqslant\ & \frac{k}{\alpha\,|s+1|} \left(h^{s+1}(b) - h^{s+1}(a)\right)^{\frac{\alpha}{k}} \int_a^b |f(t)|\, dt \\ \leqslant\ & \frac{k}{\alpha\,|s+1|} \left(h^{s+1}(b) - h^{s+1}(a)\right)^{\frac{\alpha}{k}} \|f\|_{L^1[a,b]} < \infty. \end{aligned}$$

Thus, the function T_1 is integrable over $[a,b] \times [a,b]$ by Tonelli Theorem. Hence, by Fubini theorem, we deduce that

$$\int_a^b T_1(t,\tau)\, f(t)\, dt$$

is in the space $L^1([a,b])$. Therefore, ${}_{k}^{s}J_{a,h}^{\alpha}$ $f(t)$ exists for any $t \in [a,b]$. □

Using Definitions 5 and 6, we prove the following result:

Proposition 2.9. *We have:*

$$\lim_{s \longrightarrow -1^+} {}_{k}^{s}J_{a,h}^{\alpha} f = {}_{k}I_{a,h}^{\alpha} f. \tag{9}$$

Proof. For any $t \in [a,b]$, we can write:

$$
\begin{aligned}
& \lim_{s \longrightarrow -1^{+}} {}_{k}^{s}J_{a,h}^{\alpha}\left(f\left(t\right)\right) \\
= \quad & \lim_{s \longrightarrow -1^{+}} \frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_{k}\left(\alpha\right)} \int_{a}^{t} \left(h^{s+1}\left(t\right)-h^{s+1}\left(\tau\right)\right)^{\frac{\alpha}{k}-1} h^{s}\left(\tau\right) h'\left(\tau\right) f\left(\tau\right) d\tau \\
= \quad & \lim_{s \longrightarrow -1^{+}} \frac{1}{k\Gamma_{k}\left(\alpha\right)} \int_{a}^{t} \left(\frac{h^{s+1}\left(t\right)-h^{s+1}\left(\tau\right)}{s+1}\right)^{\frac{\alpha}{k}-1} h^{s}\left(\tau\right) h'\left(\tau\right) f\left(\tau\right) d\tau \\
= \quad & \frac{1}{k\Gamma_{k}\left(\alpha\right)} \int_{a}^{t} \lim_{s \longrightarrow -1^{+}} \left(\frac{h^{s+1}\left(t\right)-h^{s+1}\left(\tau\right)}{s+1}\right)^{\frac{\alpha}{k}-1} h^{s}\left(\tau\right) h'\left(\tau\right) f\left(\tau\right) d\tau \\
= \quad & \frac{1}{k\Gamma_{k}\left(\alpha\right)} \int_{a}^{t} \left(\log \frac{h\left(t\right)}{h\left(\tau\right)}\right)^{\frac{\alpha}{k}-1} \frac{h'\left(\tau\right)}{h\left(\tau\right)} f\left(\tau\right) d\tau.
\end{aligned}
$$

Hence, the proposition is proved. □

With the same arguments as before, we can confirm that

Theorem 2.10 *The* ${}_{k}I_{\ a,h}^{\ \alpha} f(t)$ *exists for any* $t \in [a,b]$.

Now, we give the semi group properties of the $(k,s,h)-$Riemann–Liouville fractional integral with respect to h as follows:

Theorem 2.11. *Let* f *be continuous on* $[a,b]$, $k>0$, $s \in \mathbb{R}\backslash\{-1\}$, *and let* $h\left(x\right)$ *be an increasing and positive monotone function on* $[a,b]$, *having a contunuous derivative* $h'\left(x\right)$ *on* (a,b). *Then,*

$$
{}_{k}^{s}J_{a,h}^{\alpha}\left({}_{k}^{s}J_{a,h}^{\beta}\left(f\left(t\right)\right)\right) = {}_{k}^{s}J_{a,h}^{\alpha+\beta}\left(f\left(t\right)\right) = {}_{k}^{s}J_{a,h}^{\beta}\left({}_{k}^{s}J_{a,h}^{\alpha}\left(f\left(t\right)\right)\right), \tag{10}
$$

for all $\alpha,\ \beta>0,\ 0<a<t\leqslant b$.

Proof. By definition, we have

$$
\begin{aligned}
& {}_{k}^{s}J_{a,h}^{\alpha}\left({}_{k}^{s}J_{a,h}^{\beta}\left(f\left(t\right)\right)\right) \\
&= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\, h'(\tau)\ {}_{k}^{s}J_{a,h}^{\beta}\left[f(\tau)\right] d\tau \\
&= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\, h'(\tau) \qquad (11) \\
&= \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^t h^s(r)\, h'(r)\, f(r) \left[\int_r^t \left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\, h'(\tau)\right. \\
&\quad \left. \times \left(h^{s+1}(\tau)-h^{s+1}(r)\right)^{\frac{\beta}{k}-1} d\tau\right] dr.
\end{aligned}
$$

Using the change of variable

$$
x = \frac{h^{s+1}(\tau)-h^{s+1}(r)}{h^{s+1}(t)-h^{s+1}(r)}, \tag{12}
$$

we get

$$
\begin{aligned}
& \int_r^t \left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \left(h^{s+1}(\tau)-h^{s+1}(r)\right)^{\frac{\beta}{k}-1} h^s(\tau)\, h'(\tau)\, d\tau \\
&= \frac{\left(h^{s+1}(t)-h^{s+1}(r)\right)^{\frac{\alpha+\beta}{k}-1}}{s+1} \int_0^1 (1-x)^{\frac{\alpha}{k}-1} x^{\frac{\beta}{k}-1} dx \qquad (13) \\
&= \frac{k\ \left(h^{s+1}(t)-h^{s+1}(r)\right)^{\frac{\alpha+\beta}{k}-1}}{s+1} B_k(\alpha,\beta).
\end{aligned}
$$

Therefore, by (11), (13) and $k-$beta function, we have

$$
\begin{aligned}
& {}_{k}^{s}J_{a,h}^{\alpha}\left({}_{k}^{s}J_{a,h}^{\beta}\left(f\left(t\right) \right) \right) \qquad (14) \\
= & \frac{\left(s+1\right) ^{1-\frac{\alpha+\beta}{k}}}{k\Gamma_{k}\left(\alpha+\beta\right) }\int_{a}^{t}\left(h^{s+1}\left(t\right) -h^{s+1}\left(r\right) \right) ^{\frac{\alpha+\beta}{k}-1}h^{s}\left(r\right) h^{\prime}\left(r\right) f\left(r\right) dr \\
= & {}_{k}^{s}J_{a,h}^{\alpha+\beta}\left(f\left(t\right) \right) .
\end{aligned}
$$

The proof of Theorem 2.11 is completed. □

In the following result, we shall prove that the $(k,s,h)-$Hadamard integral operator is well defined. We have:

Theorem 2.12. *The* ${}_{k}^{s}I\ {}_{a,h}^{\alpha}\ f(t)$ *exists for any* $t\in[a,b]$.

Proof. Let us consider the application $T_3:[a,b]\times[a,b]\rightarrow\mathbb{R}$, such that

$$
\begin{aligned}
T_{3}\left(t,\tau\right) = & \left\lfloor \left(\log^{s+1}h\left(t\right) -\log^{s+1}h\left(\tau\right) \right) ^{\frac{\alpha}{k}-1}\log^{s}h\left(\tau\right) \frac{h^{\prime}\left(\tau\right) }{h\left(\tau\right) }\right\rfloor _{+} \qquad (15) \\
= & \left\{
\begin{array}{ll}
\left(\log^{s+1}h\left(t\right) -\log^{s+1}h\left(\tau\right) \right) ^{\frac{\alpha}{k}-1}\log^{s}h\left(\tau\right) \frac{h^{\prime}\left(\tau\right) }{h\left(\tau\right) }, & a\leqslant\tau<t\leqslant b \\
0, & ..a\leqslant t<\tau\leqslant b.
\end{array}
\right.
\end{aligned}
$$

We have T_3 is measurable on $[a,b]\times[a,b]$. Hence, we can write

$$
\begin{aligned}
& \left\vert \int_{a}^{b}\left(\int_{a}^{b}\left(\log^{s+1}h\left(t\right) -\log^{s+1}h\left(\tau\right) \right) ^{\frac{\alpha}{k}-1}\log^{s}h\left(\tau\right) \frac{h^{\prime}\left(\tau\right) }{h\left(\tau\right) }f\left(\tau\right) d\tau\right) dt\right\vert \\
\leqslant & \int_{a}^{b}\left\vert f\left(t\right) \right\vert \left(\left\vert \int_{a}^{t}\left(\log^{s+1}h\left(t\right) -\log^{s+1}h\left(\tau\right) \right) ^{\frac{\alpha}{k}-1}\log^{s}h\left(\tau\right) \frac{h^{\prime}\left(\tau\right) }{h\left(\tau\right) }\right\vert d\tau\right) dt \\
\leqslant & \frac{k}{\alpha\left\vert s+1\right\vert }\int_{a}^{b}\left(\log^{s+1}h\left(t\right) -\log^{s+1}h\left(a\right) \right) ^{\frac{\alpha}{k}}\left\vert f\left(t\right) \right\vert dt \qquad (16) \\
\leqslant & \frac{k\left(\log^{s+1}h\left(b\right) -\log^{s+1}h\left(a\right) \right) ^{\frac{\alpha}{k}}}{\alpha\left\vert s+1\right\vert }\int_{a}^{b}\left\vert f\left(t\right) \right\vert dt \\
\leqslant & \frac{k\left(\log^{s+1}h\left(b\right) -\log^{s+1}h\left(a\right) \right) ^{\frac{\alpha}{k}}}{\alpha\left\vert s+1\right\vert }\left\Vert f\right\Vert _{L^{1}\left[a,b\right] }<\infty.
\end{aligned}
$$

Consequently, T_3 is integrable over $[a,b]\times[a,b]$ and

$$\int_a^b T_3\left(t,\tau\right)f\left(t\right)dt$$

is an integrable on $[a,b]$. That is ${}_k^sI_{\ a,h}^{\ \alpha}\ f(t)$ exists for any $t\in[a,b]$. □

Theorem 2.13. *Let g be an increasing, positive, monotone function with $g\in C^1([a,b])$. If $h\left(t\right)=\ln g\left(t\right)$ over $[a,b]$, then*

$${}_kJ_{a,h}^{\alpha}f={}_kI_{a,g}^{\alpha}f,\ \textit{and}\ {}_k^sJ_{a,h}^{\alpha}f={}_k^sI_{a,g}^{\alpha}f.$$

Proof. By Definition 3, we have

$$\begin{aligned}
{}_kJ_{a,h}^{\alpha}f(t) &= \frac{1}{k\Gamma_k\left(\alpha\right)}\int_a^t\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}h'\left(\tau\right)f\left(\tau\right)d\tau\\
&= \frac{1}{k\Gamma_k\left(\alpha\right)}\int_a^t\left(\ln g\left(t\right)-\ln g\left(\tau\right)\right)^{\frac{\alpha}{k}-1}\frac{g'\left(\tau\right)}{g\left(\tau\right)}f\left(\tau\right)d\tau\\
&= {}_kI_{a,g}^{\alpha}f(t).
\end{aligned}$$

On the other hand, we observe that

$$\begin{aligned}
{}_k^sJ_{a,h}^{\alpha}f\left(t\right) &= \frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_k\left(\alpha\right)}\int_a^t\left(h^{s+1}\left(t\right)-h^{s+1}\left(\tau\right)\right)^{\frac{\alpha}{k}-1}h^s\left(\tau\right)h'\left(\tau\right)d\tau\\
&= \frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_k\left(\alpha\right)}\int_a^t\left(\ln^{s+1}g\left(t\right)-\ln^{s+1}g\left(\tau\right)\right)^{\frac{\alpha}{k}-1}\ln^s g\left(\tau\right)\frac{g'\left(\tau\right)}{g\left(\tau\right)}d\tau\\
&= {}_k^sI_{a,g}^{\alpha}f(t).
\end{aligned}$$

The proof is completed. □

Corollary 2.14. *Let $k>0,\ \alpha>0$ and $s\in\mathbb{R}\backslash\left\{-1\right\}$. Then, we have*

$$\begin{aligned}
{}_k^sI_{a,g}^{\alpha}\left(1\right) &= \frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_k\left(\alpha\right)}\int_a^t\left(\log^{s+1}g\left(t\right)-\log^{s+1}g\left(\tau\right)\right)^{\frac{\alpha}{k}-1}\log^s g\left(\tau\right)\frac{g'\left(\tau\right)}{g\left(\tau\right)}d\tau\\
&= \frac{1}{\left(s+1\right)^{\frac{\alpha}{k}}\Gamma_k\left(\alpha+k\right)}\left(\log^{s+1}g\left(t\right)-\log^{s+1}g\left(a\right)\right)^{\frac{\alpha}{k}},\alpha>0. \qquad (17)
\end{aligned}$$

Now we present to the reader the semi group and the commutativity properties for the $(k,s,h)-$ Hadamard integral operator:

Theorem 2.15. *Let f be continuous on $[a,b]$, $k>0$, $s\in\mathbb{R}\backslash\{-1\}$, and let $g(x)$ be an increasing and positive monotone function on $[a,b]$, having a contunuous derivative $g'(x)$ on (a,b). Then, we have*

$$ {}_{k}^{s}I_{a,g}^{\alpha}\left({}_{k}^{s}I_{a,g}^{\beta}\left(f\left(t\right)\right)\right)={}_{k}^{s}I_{a,g}^{\alpha+\beta}\left(f\left(t\right)\right)={}_{k}^{s}I_{a,g}^{\beta}\left({}_{k}^{s}I_{a,g}^{\alpha}\left(f\left(t\right)\right)\right), \quad (18)$$

where $\alpha,\ \beta>0,\ 0<a<t\leqslant b$.

Proof. We have

$$\begin{aligned}
&{}_{k}^{s}I_{a,g}^{\alpha}\left({}_{k}^{s}I_{a,g}^{\beta}\left(f\left(t\right)\right)\right)\\
&=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(\log^{s+1}g(t)-\log^{s+1}g(\tau)\right)^{\frac{\alpha}{k}-1}\log^s g(\tau)\frac{g'(\tau)}{g(\tau)}\,{}_{k}^{s}I_{a,g}^{\beta}\left[f(\tau)\right]d\tau\\
&=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(\log^{s+1}g(t)-\log^{s+1}g(\tau)\right)^{\frac{\alpha}{k}-1}\log^s g(\tau)\frac{g'(\tau)}{g(\tau)} \quad (19)\\
&\times\left[\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^\tau\left(\log^{s+1}g(\tau)-\log^{s+1}g(r)\right)^{\frac{\alpha}{k}-1}\log^s g(r)\frac{g'(r)}{g(r)}f(r)\,dr\right]d\tau\\
&=\frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\log^s g(r)\frac{g'(r)}{g(r)}f(r)\\
&\times\left[\int_r^t\left(\log^{s+1}g(t)-\log^{s+1}g(\tau)\right)^{\frac{\alpha}{k}-1}\log^s g(\tau)\frac{g'(\tau)}{g(\tau)}\right.\\
&\times\left.\left(\log^{s+1}g(\tau)-\log^{s+1}g(r)\right)^{\frac{\beta}{k}-1}d\tau\right]dr.
\end{aligned}$$

Thanks to the change of variable

$$x=\frac{\log^{s+1}g(\tau)-\log^{s+1}g(r)}{\log^{s+1}g(t)-\log^{s+1}g(r)}, \quad (20)$$

it yields that

$$
\begin{aligned}
&\int_r^t \left(\log^{s+1} g(t) - \log^{s+1} g(\tau)\right)^{\frac{\alpha}{k}-1} \left(\log^{s+1} g(\tau) - \log^{s+1} g(r)\right)^{\frac{\beta}{k}-1} \log^s g(\tau) \frac{g'(\tau)}{g(\tau)} d\tau \\
= \quad & \frac{\left(\log^{s+1} g(t) - \log^{s+1} g(r)\right)^{\frac{\alpha+\beta}{k}-1}}{s+1} \int_0^1 (1-x)^{\frac{\alpha}{k}-1} x^{\frac{\beta}{k}-1} dx \\
= \quad & \frac{k\ \left(\log^{s+1} g(t) - \log^{s+1} g(r)\right)^{\frac{\alpha+\beta}{k}-1}}{s+1} B_k(\alpha,\beta).
\end{aligned} \tag{21}
$$

Therefore, by $(19), (21)$ and by the $k-$Beta function, we obtain

$$
\begin{aligned}
& {}_k^sI_{a,g}^{\alpha}\left({}_k^sI_{a,g}^{\beta}\left(f(t)\right)\right) \\
= \quad & \frac{(s+1)^{1-\frac{\alpha+\beta}{k}}}{k\Gamma_k(\alpha+\beta)} \int_a^t \left(\log^{s+1} g(t) - \log^{s+1} g(r)\right)^{\frac{\alpha+\beta}{k}-1} \log^s g(r) \frac{g'(r)}{g(r)} f(r)\, dr \\
= \quad & {}_k^sI_{a,g}^{\alpha+\beta}\left(f(t)\right).
\end{aligned} \tag{22}
$$

Theorem 2.15 is thus proved. □

3. Applications

Theorem 3.1. *Let f and g be two synchronous functions on $[0,\infty)$ and let h be an increasing and positive monotone function on $[a,b]$, having a contunuous derivative $h'(u)$ on (a,b). Then for $0 < a < t < \infty$ and $\alpha > 0$, the following inequality holds:*

$$
{}_k^sI_{a,h}^{\alpha}\left[(fg)(t)\right] \geqslant \frac{1}{{}_k^sI_{a,h}^{\alpha}(1)}\ {}_k^sI_{a,h}^{\alpha}\left[f(t)\right]\ {}_k^sI_{a,h}^{\alpha}\left[g(t)\right]. \tag{23}
$$

Proof. Consider

$$
\begin{aligned}
{}_k^sF_h^{\alpha}(t,u) &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \left(\log^{s+1} h(t) - \log^{s+1} h(u)\right)^{\frac{\alpha}{k}-1} \log^s h(u) \frac{h'(u)}{h(u)}, s \neq -1 \\
{}_k^sF_h^{\alpha}(t,v) &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \left(\log^{s+1} h(t) - \log^{s+1} h(v)\right)^{\frac{\alpha}{k}-1} \log^s h(v) \frac{h'(v)}{h(v)}, s \neq -1.
\end{aligned}
$$

The functions f and g are synchronous on $[0,+\infty)$, so for all $u, v \geqslant 0$, we have

$$(f(u)-f(v))(g(u)-g(v)) \geqslant 0,$$

imply that

$$f(u)g(u)+f(v)g(v) \geqslant f(u)g(v)+f(v)g(u). \tag{24}$$

Multipling both sides of (24) by ${}_{k}^{s}F_{h}^{\alpha}(t,u) \times {}_{k}^{s}F_{h}^{\alpha}(t,v), u, v \in (a,t)$, and double integrating the resulting identity with respect to u and v from a to t, we obtain

$${}_{k}^{s}I_{a,h}^{\alpha}[(fg)(t)]{}_{k}^{s}I_{a,h}^{\alpha}(1) \geqslant {}_{k}^{s}I_{a,h}^{\alpha}[f(t)]\ {}_{k}^{s}I_{a,h}^{\alpha}[g(t)].$$

Theorem 3.1 is thus proved. □

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