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Strongly Quasi-Duo Rings

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Abstract. In this paper we extend the concepts of two sided ideal and right quasi-duo ring. These ideals and rings are called totally fully invariant and right strongly quasi-duo, respectively. Right strongly quasi-duo rings are always right quasi-duo. We investigate the properties of these rings and ideals and show among other things that right strongly quasi-duo rings are classical, directly finite and co-hopfian while right quasi duo, right duo or commutative rings do not have necessarily these properties.

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1. Introduction

Throughout this article, all rings are associative with identity. A noncommutative ring R is called right(left) duo if any right(left) ideal of R is a two sided ideal. These rings are extensively studied (see for example [9] and [10]). A right ideal I of R is called totally fully invariant if for each $a \in I$ and $b \in R$, $\mathrm{ann_r}(a) \subseteq \mathrm{ann_r}(b)$ implies that $b \in I$. It is clear that totally fully invariant right ideals are two sided. In [2], we studied rings whose all right ideals are totally fully invariant. We called these rings "right strongly duo". A noncommutative ring R is called right (left) quasi-duo if any maximal right (left) ideal of R is a two sided ideal. A ring R is called right (left) strongly quasi-duo if any maximal right (left) ideal of R is a totally fully invariant right ideal. Commutative rings are

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clearly left and right duo but they are not always strongly quasi-duo. Local rings are right quasi-duo because Jacobson radical of any ring is two sided ideal but they are not always right strongly quasi-duo, for example, let P be a prime ideal of commutative domain R. Then R_P is a local domain and hence it has no nontrivial totally fully invariant ideal. Throughout this article, by $\mathrm{J}(R)$ and $Z(R)$ we mean Jacobson radical and singular ideal of R, respectively. In Section 2, we study totally fully invariant right ideals and their equivalent conditions. In Section 3, we study right strongly quasi-duo rings. All unexplained terminologies and basic results on rings that are used in the sequel can be found in [1], [5], [6] and [7].

2. Totally Fully Invariant Right Ideals

A right ideal I of R is called totally fully invariant provided for each $a \in I$ and $b \in R$, $\mathrm{ann_r}(a) \subseteq \mathrm{ann_r}(b)$ implies that $b \in I$, and it is denoted by $I \trianglelefteq_t R$. For any ring R, 0 and R are totally fully invariant right ideal, clearly. If R is a domain, then totally fully invariant right ideals are precisely trivial ideals and the converse is true if R is a commutative ring.

Example 2.1. Every ideal of $\mathbb{Z}_n$ is totally fully invariant, where $n \geqslant 2$. Commutative rings are duo but they are not always strongly quasi duo. For example, $\mathbb{Z}$ is a commutative domain, hence it has no nontrivial totally fully invariant ideal.

Lemma 2.2. *Every totally fully invariant right ideal of R is a two sided ideal.*

Proof. Two sided ideals of a ring are precisely fully invariant right ideals. Let I be a totally fully invariant right ideal of R and $f : R \longrightarrow R$ be an R- homomorphism. For each $a \in I$, $\mathrm{ann_r}(a) \subseteq \mathrm{ann_r}(f(a))$. Since I is a totally fully invariant right ideal, then $f(a) \in I$. □

By [8], a ring R is called right *principally injective* ring, if for each principal right ideal I of R, any R-homomorphism $f \in \mathrm{Hom}_R(I, R)$ can be extended to an R-homomorphism $\overline{f} \in \mathrm{Hom}_R(R, R)$.

Lemma 2.3. *Let R be a right principally injective ring. Then totally fully invariant right ideals of R are precisely fully invariant right ideals.*

Proof. By Lemma 2.2, it is sufficient to show that fully invariant right ideals are totally fully invariant. Let I be a fully invariant right ideal of R, $a \in I$ and $b \in R$ such that $\mathrm{ann_r}(a) \subseteq \mathrm{ann_r}(b)$. Then the map $f : aR \longrightarrow R$ with $f(ar) = br$ for each $r \in R$, is an R-homomorphism. Since R is a right principally injective ring, the R-homomorphism f can be extended to $g \in \mathrm{End}(R_R)$. Therefore $g(I) \subseteq I$ because I is a fully invariant right ideal, and hence $b = f(a) = g(a) \in I$. □

Proposition 2.4. *A right ideal I of R is totally fully invariant if and only if for each right ideal J of R contained in I and each $f \in \mathrm{Hom}_R(J, R)$, $f(J) \subseteq I$.*

Proof. Let I be a totally fully invariant right ideal of R, $J \subseteq I$ and $f \in \mathrm{Hom}_R(J, R)$. For each $a \in J$, $\mathrm{ann_r}(a) \subseteq \mathrm{ann_r}(f(a))$ implies that $f(a) \in I$ because I is a totally fully invariant right ideal. Conversely, assume that $a \in I$ and $\mathrm{ann_r}(a) \subseteq \mathrm{ann_r}(b)$ for some $b \in R$. Therefore the map $f : aR \longrightarrow R$ with $f(ar) = br$ ($\forall r \in R$) is an R-homomorphism. By assumption, $b = f(a) \in f(aR) \subseteq I$. □

Definition 2.5. *For each right ideal I of R, define*

$$C_T(I) = \sum\{A|\ A \trianglelefteq_t R \text{ and } A \cap I = 0\}.$$

It is easy to show that $C_T(I) = \{x \in R|\ \forall\ 0 \neq a \in I \text{ and } \forall\ r \in R,\ \mathrm{ann_r}(xr) \nsubseteq \mathrm{ann_r}(a)\}$. It is easy to see that $C_T(I)$ is a totally fully invariant right ideal of R such that $C_T(I) \cap I = 0$.

3. Strongly Quasi-duo Rings

In this section we investigate the strongly quasi-duo rings and their properties. A ring R is called right quasi-duo ring provided every maximal right ideal of R is a two sided. First we state the following equivalent assertion to this concept.

Proposition 3.1. *The ring R is right quasi-duo if and only if for each maximal right ideal M of R, $Rej(R, R/M) = M$.*

Proof. The "if" part is clear because for each right ideal I of R, $Rej(R, R/I)$ is two sided ideal [1, Corollary 8.23]. Let M be a maximal right ideal of R and $0 \neq f \in \mathrm{Hom}_R(R, R/M)$. There exists $a \in R - M$ such that $f(1) = a + M$. Since M is a two sided ideal, then for each $x \in M$, $f(x) = ax + M = M$. Therefore $M \subseteq \ker f$. Hence $M \subseteq Rej(R, R/M)$ the equality is hold. □

Lemma 3.2. *Let R be a right quasi-duo ring and M be a maximal right ideal of R. Then for each $a \in R$, $a^2 \in M$ implies that $a \in M$.*

Proof. Since M is a two sided ideal, then $M \subseteq (M : a)$. Hence either $(M : a) = R$ or $(M : a) = M$. If $(M : a) = M$, then $a^2 \in M$ implies that $a \in (M : a) = M$ and hence $M = (M : a) = R$, a contradiction. Therefore $(M : a) = R$ and hence $a \in M$. □
In [3], the last part of the following proposition has been proved by a sophisticated technique, here we give a much simpler proof.

Proposition 3.3. *Let R be a right quasi-duo ring and M be a maximal right ideal of R. Then for each $a, b \in R$, $ab \in M$ implies that either $a \in M$ or $b \in M$. Moreover, all nilpotent elements of R are in $\mathrm{J}(R)$.*

Proof. Since M is a two sided ideal of R, then for each $r \in R$, $(bra)^2 = (br)(ab)(ra) \in M$. By Lemma 3.2, $bra \in M$. Since it is true for each $r \in R$, then $bRa \subseteq M$. By [5, Proposition 10.2], M is a prime ideal of R. Hence $bRa \subseteq M$ implies that either $a \in M$ or $b \in M$. □

Definition 3.4. *The ring R is called right strongly quasi-duo ring provided every maximal right ideal of R is a totally fully invariant right ideal.*

Example 3.5. For each positive integer number $n \geqslant 2$, $\mathbb{Z}_n$ is a right strongly quasi-duo ring.

Example 3.6. Any divisible right quasi-duo ring is a right strongly quasi-duo ring.

Lemma 3.7. *Right strongly quasi-duo rings are right quasi-duo.*

Proof. The verification is immediate. □

Lemma 3.8. *Let R be a right strongly quasi-duo ring. Then for each maximal right ideal M of R, either $C_T(M) = 0$ or $C_TC_T(M) = M$.*

Proof. Since M is a totally fully invariant right ideal and $C_T(M) \cap M = 0$, then $M \subseteq C_TC_T(M)$. Therefore either $C_TC_T(M) = M$ or $C_TC_T(M) = R$. Since $C_TC_T(M) \cap C_T(M) = 0$, then $C_TC_T(M) = M$ or $C_T(M) = 0$ □

Lemma 3.9. *Let R be right strongly quasi-duo ring. Then $\mathrm{J}(R)$ is a totally fully invariant right ideal of R.*

Proof. It is clear because the intersection of totally fully invariant right ideals is totally fully invariant. □

Proposition 3.10. *Let R be a right strongly quasi-duo ring. Then*

$$\mathrm{U}(R) = \{a \in R|\ \mathrm{ann_r}(a) = 0\}.$$

Proof. It is obvious that if $a \in \mathrm{U}(R)$, then $\mathrm{ann_r}(a) = 0$. Conversely, we show that if $\mathrm{ann_r}(b) = 0$ for some $b \in R$, then $bR = R$. If bR is a proper right ideal of R, there exists a maximal right ideal M containing bR. Hence $b \in M$ and for each $x \in R$, $0 = \mathrm{ann_r}(b) \subseteq \mathrm{ann_r}(x)$. Since M is a totally fully invariant right ideal of R and $b \in M$, then $x \in M$. Since it is true for each $x \in R$, then $M = R$. It is a contradiction. Thus $bR = R$. Therefore $\mathrm{ann_r}(b) = 0$ implies that $ba = 1$ for some $a \in R$. It is clear that $\mathrm{ann_r}(a) = 0$ and hence a has a right inverse, say $c \in R$. Since $b = b(ac) = (ba)c = c$, then ba=ab=1 and hence $b \in \mathrm{U}(R)$. □

By [6, 10.17], the ring R is called *classical* if $\mathrm{U}(R) = \{a \in R|\ \mathrm{ann_l}(a) = \mathrm{ann_r}(a) = 0\}$.

Corollary 3.11. *Let R be a right strongly quasi-duo ring. Then $\mathrm{Z}(R) \subseteq \mathrm{J}(R)$.*

Proof. Let $a \in \mathrm{Z}(R)$. For each $x \in R$, $\mathrm{ann_r}(a) \cap \mathrm{ann_r}(1 - xa) = 0$. Therefore $\mathrm{ann_r}(1 - xa) = 0$. Hence by Proposition 3.10, $1 - xa \in \mathrm{U}(R)$.

It implies that $a \in \mathrm{J}(R)$. $\square$

Corollary 3.12. *R is a division ring if and only if R is a right strongly quasi-duo domain.*

Proof. If R is a division ring, then 0 is the unique maximal right ideal of R. Hence R is a right strongly quasi-duo domain. Conversely, assume that R is a right strongly quasi-duo domain. Since R is a domain, then $\mathrm{ann_r}(a) = 0$ for each $0 \neq a \in R$. Hence by Proposition 3.10,

$$\mathrm{U}(R) = \{a \in R |\ \mathrm{ann_r}(a) = 0\} = R - \{0\}.$$

Therefore R is a division ring. $\square$

Theorem 3.13. *Let R be a hereditary and right strongly quasi-duo ring. Then* $\frac{R}{\mathrm{J}(R)}$ *is a right strongly quasi-duo ring.*

Proof. Let $M/\mathrm{J}(R)$ be a right maximal ideal of $R/\mathrm{J}(R)$, $N/\mathrm{J}(R)$ be a right ideal of $R/\mathrm{J}(R)$ contained in $M/\mathrm{J}(R)$ and $f \in \mathrm{Hom}_{\frac{R}{\mathrm{J}(R)}}(N/\mathrm{J}(R), R/\mathrm{J}(R))$. It is clear that f is an R-homomorphism. Let $p : N \longrightarrow N/\mathrm{J}(R)$ and $\pi : R \longrightarrow R/\mathrm{J}(R)$ be projection maps. Since N is a projective as an R-module, there exists an R-homomorphism $g : N \longrightarrow R$ such that $\pi g = fp$. Since M is a totally fully invariant right ideal of R, then by Proposition 2.4, $g(N) \subseteq M$. Thus

$$f(N/\mathrm{J}(R)) = fp(N) = \pi g(N) \subseteq \pi(M) \subseteq M/\mathrm{J}(R).$$

Therefore by Proposition 2.4, $M/\mathrm{J}(R)$ is a totally fully invariant right ideal of $R/\mathrm{J}(R)$. $\square$

Theorem 3.14. *Let R be a right strongly quasi-duo ring and I be a proper right ideal of R. Then* $\mathrm{Hom}_R(I, R)$ *has no epimorphism and* $\mathrm{Hom}_R(R, I)$ *has no monomorphism element.*

Proof. Let I be a proper right ideal of R, M be a maximal right ideal of R containing I and $f \in \mathrm{Hom}_R(I, R)$ be an epimorphism. For each $b \in R$ there exists $a \in I$ such that $f(a) = b$. Then by Proposition 4, $b \in f(I) \subseteq M$ and hence $R \subseteq M$. It is a contradiction. Again, Let I be a proper right ideal of R, M be a maximal right ideal of R containing

I and $f \in \text{Hom}_R(R, I)$ be a monomorphism. since f is monic, then for each $a \in R$, $\text{ann}_\text{r}(a) = \text{ann}_\text{r}(f(a))$. Since M is a totally fully invariant right ideal of R and $f(a) \in M$, then $\text{ann}_\text{r}(a) = \text{ann}_\text{r}(f(a))$ implies that $a \in M$. Hence $R \subseteq M$. It is a contradiction. □

A ring R is called *directly finite* if R is not isomorphic to a proper summand of itself. By [7, Proposition 1.25], R is directly finite if and only if for each $a, b \in R$, $ab = 1$ implies that $ba = 1$.

Corollary 3.15. *If R a right strongly quasi-duo ring, then R is not isomorphic to any proper right ideal of itself. In particular, R is a directly finite ring.*

Proof. It is clear by Theorem 3.14. □

Corollary 3.16. *Right strongly quasi-duo rings are co-hopfian.*

Proof. Let $f \in \text{End}(R_R)$ be a monomorphism. Then $R \cong \text{Im }(f)$. Therefore by Corollary 3.15, $R = \text{Im }(f)$. □

Proposition 3.17. *If R is a right strongly quasi-duo ring, then nonequal maximal right ideals of R are not isomorphic.*

Proof. Let M_1 and M_2 be two maximal right ideals of R which are isomorphic. Then there exists an R-isomorphism $f \in Hom_R(M_1, M_2)$. For each $a \in M_1$, $\text{ann}_\text{r}(a) = \text{ann}_\text{r}(f(a))$. Since M_2 is a totally fully invariant right ideal of R and $f(a) \in M_2$, then $\text{ann}_\text{r}(f(a)) \subseteq \text{ann}_\text{r}(a)$ implies that $a \in M_2$. Therefore $M_1 \subseteq M_2$ and hence $M_1 = M_2$ □

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