Strongly Quasi-Duo Rings

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Abstract. In this paper we extend the concepts of two sided ideal and right quasi-duo ring. These ideals and rings are called totally fully invariant and right strongly quasi-duo, respectively. Right strongly quasi-duo rings are always right quasi-duo. We investigate the properties of these rings and ideals and show among other things that right strongly quasi-duo rings are classical, directly finite and co-hopfian while right quasi duo, right duo or commutative rings do not have necessarily these properties.

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1. Introduction

Throughout this article, all rings are associative with identity. A non-commutative ring $R$ is called right (left) duo if any right (left) ideal of $R$ is a two sided ideal. These rings are extensively studied (see for example [9] and [10]). A right ideal $I$ of $R$ is called totally fully invariant if for each $a \in I$ and $b \in R$, $\text{ann}_r(a) \subseteq \text{ann}_r(b)$ implies that $b \in I$. It is clear that totally fully invariant right ideals are two sided. In [2], we studied rings whose all right ideals are totally fully invariant. We called these rings ”right strongly duo”. A noncommutative ring $R$ is called right (left) quasi-duo if any maximal right (left) ideal of $R$ is a two sided ideal. A ring $R$ is called right (left) strongly quasi-duo if any maximal right (left) ideal of $R$ is a totally fully invariant right ideal. Commutative rings are
clearly left and right duo but they are not always strongly quasi-duo. Local rings are right quasi-duo because Jacobson radical of any ring is two sided ideal but they are not always right strongly quasi-duo; for example, let $P$ be a prime ideal of commutative domain $R$. Then $RP$ is a local domain and hence it has no nontrivial totally fully invariant ideal. Throughout this article, by $J(R)$ and $Z(R)$ we mean Jacobson radical and singular ideal of $R$, respectively. In Section 2, we study totally fully invariant right ideals and their equivalent conditions. In Section 3, we study right strongly quasi-duo rings. All unexplained terminologies and basic results on rings that are used in the sequel can be found in [1], [5], [6] and [7].

2. Totally Fully Invariant Right Ideals

A right ideal $I$ of $R$ is called totally fully invariant provided for each $a \in I$ and $b \in R$, $\text{ann}_r(a) \subseteq \text{ann}_r(b)$ implies that $b \in I$, and it is denoted by $I \trianglelefteq^t R$. For any ring $R$, 0 and $R$ are totally fully invariant right ideal, clearly. If $R$ is a domain, then totally fully invariant right ideals are precisely trivial ideals and the converse is true if $R$ is a commutative ring.

Example 2.1. Every ideal of $\mathbb{Z}_n$ is totally fully invariant, where $n \geq 2$. Commutative rings are duo but they are not always strongly quasi duo. For example, $\mathbb{Z}$ is a commutative domain, hence it has no nontrivial totally fully invariant ideal.

Lemma 2.2. Every totally fully invariant right ideal of $R$ is a two sided ideal.

Proof. Two sided ideals of a ring are precisely fully invariant right ideals. Let $I$ be a totally fully invariant right ideal of $R$ and $f : R \to R$ be an $R$-homomorphism. For each $a \in I$, $\text{ann}_r(a) \subseteq \text{ann}_r(f(a))$. Since $I$ is a totally fully invariant right ideal, then $f(a) \in I$. □

By [8], a ring $R$ is called right principally injective ring, if for each principal right ideal $I$ of $R$, any $R$-homomorphism $f \in \text{Hom}_R(I,R)$ can be extended to an $R$-homomorphism $\overline{f} \in \text{Hom}_R(R,R)$.
Lemma 2.3. Let $R$ be a right principally injective ring. Then totally fully invariant right ideals of $R$ are precisely fully invariant right ideals.

Proof. By Lemma 2.2, it is sufficient to show that fully invariant right ideals are totally fully invariant. Let $I$ be a fully invariant right ideal of $R$, $a \in I$ and $b \in R$ such that $\text{ann}_r(a) \subseteq \text{ann}_r(b)$. Then the map $f : aR \to R$ with $f(ar) = br$ for each $r \in R$, is an $R$-homomorphism. Since $R$ is a right principally injective ring, the $R$-homomorphism $f$ can be extended to $g \in \text{End}(R_R)$. Therefore $g(I) \subseteq I$ because $I$ is a fully invariant right ideal, and hence $b = f(a) = g(a) \in I$. □

Proposition 2.4. A right ideal $I$ of $R$ is totally fully invariant if and only if for each right ideal $J$ of $R$ contained in $I$ and each $f \in \text{Hom}_R(J, R)$, $f(J) \subseteq I$.

Proof. Let $I$ be a totally fully invariant right ideal of $R$, $J \subseteq I$ and $f \in \text{Hom}_R(J, R)$. For each $a \in J$, $\text{ann}_r(a) \subseteq \text{ann}_r(f(a))$ implies that $f(a) \in I$ because $I$ is a totally fully invariant right ideal. Conversely, assume that $a \in I$ and $\text{ann}_r(a) \subseteq \text{ann}_r(b)$ for some $b \in R$. Therefore the map $f : aR \to R$ with $f(ar) = br$ $(\forall r \in R)$ is an $R$-homomorphism. By assumption, $b = f(a) \in f(aR) \subseteq I$. □

Definition 2.5. For each right ideal $I$ of $R$, define

$$C_T(I) = \sum \{ A | A \leq_R R \text{ and } A \cap I = 0 \}.$$

It is easy to show that $C_T(I) = \{ x \in R | \forall 0 \neq a \in I \text{ and } \forall r \in R, \text{ann}_r(xr) \not\subseteq \text{ann}_r(a) \}$. It is easy to see that $C_T(I)$ is a totally fully invariant right ideal of $R$ such that $C_T(I) \cap I = 0$.

3. Strongly Quasi-duo Rings

In this section we investigate the strongly quasi-duo rings and their properties. A ring $R$ is called right quasi-duo ring provided every maximal right ideal of $R$ is a two sided. First we state the following equivalent assertion to this concept.
Proposition 3.1. The ring $R$ is right quasi-duo if and only if for each maximal right ideal $M$ of $R$, $\text{Rej}(R, R/M) = M$.

Proof. The "if" part is clear because for each right ideal $I$ of $R$, $\text{Rej}(R, R/I)$ is two sided ideal [1, Corollary 8.23]. Let $M$ be a maximal right ideal of $R$ and $0 \neq f \in \text{Hom}_R(R, R/M)$. There exists $a \in R - M$ such that $f(1) = a + M$. Since $M$ is a two sided ideal, then for each $x \in M$, $f(x) = ax + M = M$. Therefore $M \subseteq \ker f$. Hence $M \subseteq \text{Rej}(R, R/M)$ the equality is hold. \qed

Lemma 3.2. Let $R$ be a right quasi-duo ring and $M$ be a maximal right ideal of $R$. Then for each $a \in R$, $a^2 \in M$ implies that $a \in M$.

Proof. Since $M$ is a two sided ideal, then $M \subseteq (M : a)$. Hence either $(M : a) = R$ or $(M : a) = M$. If $(M : a) = M$, then $a^2 \in M$ implies that $a \in (M : a) = M$ and hence $M = (M : a) = R$, a contradiction. Therefore $(M : a) = R$ and hence $a \in M$. \qed

In [3], the last part of the following proposition has been proved by a sophisticated technique, here we give a much simpler proof.

Proposition 3.3. Let $R$ be a right quasi-duo ring and $M$ be a maximal right ideal of $R$. Then for each $a, b \in R$, $ab \in M$ implies that either $a \in M$ or $b \in M$. Moreover, all nilpotent elements of $R$ are in $J(R)$.

Proof. Since $M$ is a two sided ideal of $R$, then for each $r \in R$, $(br)a^2 = (br)(ab)(ra) \in M$. By Lemma 3.2, $bra \in M$. Since it is true for each $r \in R$, then $bRa \subseteq M$. By [5, Proposition 10.2], $M$ is a prime ideal of $R$. Hence $bRa \subseteq M$ implies that either $a \in M$ or $b \in M$. \qed

Definition 3.4. The ring $R$ is called right strongly quasi-duo ring provided every maximal right ideal of $R$ is a totally fully invariant right ideal.

Example 3.5. For each positive integer number $n \geq 2$, $\mathbb{Z}_n$ is a right strongly quasi-duo ring.

Example 3.6. Any divisible right quasi-duo ring is a right strongly quasi-duo ring.
Lemma 3.7. Right strongly quasi-duo rings are right quasi-duo.

Proof. The verification is immediate. □

Lemma 3.8. Let \( R \) be a right strongly quasi-duo ring. Then for each maximal right ideal \( M \) of \( R \), either \( C_T(M) = 0 \) or \( C_TC_T(M) = M \).

Proof. Since \( M \) is a totally fully invariant right ideal and \( C_TC_T(M) \cap M = 0 \), then \( M \subseteq C_TC_T(M) \). Therefore either \( C_TC_T(M) = M \) or \( C_TC_T(M) = R \). Since \( C_TC_T(M) \cap C_T(M) = 0 \), then \( C_TC_T(M) = M \) or \( C_T(M) = 0 \). □

Lemma 3.9. Let \( R \) be right strongly quasi-duo ring. Then \( J(R) \) is a totally fully invariant right ideal of \( R \).

Proof. It is clear because the intersection of totally fully invariant right ideals is totally fully invariant. □

Proposition 3.10. Let \( R \) be a right strongly quasi-duo ring. Then

\[
U(R) = \{a \in R| \ann_r(a) = 0\}.
\]

Proof. It is obvious that if \( a \in U(R) \), then \( \ann_r(a) = 0 \). Conversely, we show that if \( \ann_r(b) = 0 \) for some \( b \in R \), then \( bR = R \). If \( bR \) is a proper right ideal of \( R \), there exists a maximal right ideal \( M \) containing \( bR \). Hence \( b \in M \) and for each \( x \in R \), \( 0 = \ann_r(b) \subseteq \ann_r(x) \). Since \( M \) is a totally fully invariant right ideal of \( R \) and \( b \in M \), then \( x \in M \). Since it is true for each \( x \in R \), then \( M = R \). It is a contradiction. Thus \( bR = R \). Therefore \( \ann_r(b) = 0 \) implies that \( ba = 1 \) for some \( a \in R \). It is clear that \( \ann_r(a) = 0 \) and hence \( a \) has a right inverse, say \( c \in R \). Since \( b = b(ac) = (ba)c = c \), then \( ba = ab = 1 \) and hence \( b \in U(R) \). □

By [6, 10.17], the ring \( R \) is called classical if \( U(R) = \{a \in R| \ann_l(a) = \ann_r(a) = 0\} \).

Corollary 3.11. Let \( R \) be a right strongly quasi-duo ring. Then \( Z(R) \subseteq J(R) \).

Proof. Let \( a \in Z(R) \). For each \( x \in R \), \( \ann_r(a) \cap \ann_r(1 - xa) = 0 \). Therefore \( \ann_r(1 - xa) = 0 \). Hence by Proposition 3.10, \( 1 - xa \in U(R) \).
It implies that \( a \in J(R) \). □

**Corollary 3.12.** \( R \) is a division ring if and only if \( R \) is a right strongly quasi-duo domain.

**Proof.** If \( R \) is a division ring, then 0 is the unique maximal right ideal of \( R \). Hence \( R \) is a right strongly quasi-duo domain. Conversely, assume that \( R \) is a right strongly quasi-duo domain. Since \( R \) is a domain, then \( \text{ann}_r(a) = 0 \) for each \( 0 \neq a \in R \). Hence by Proposition 3.10,

\[
U(R) = \{ a \in R \mid \text{ann}_r(a) = 0 \} = R - \{ 0 \}.
\]

Therefore \( R \) is a division ring. □

**Theorem 3.13.** Let \( R \) be a hereditary and right strongly quasi-duo ring. Then \( \frac{R}{J(R)} \) is a right strongly quasi-duo ring.

**Proof.** Let \( M/J(R) \) be a right maximal ideal of \( R/J(R) \), \( N/J(R) \) be a right ideal of \( R/J(R) \) contained in \( M/J(R) \) and \( f \in \text{Hom}_{\frac{R}{J(R)}}(N/J(R), R/J(R)) \). It is clear that \( f \) is an \( R \)-homomorphism. Let \( p : N \to N/J(R) \) and \( \pi : R \to R/J(R) \) be projection maps. Since \( N \) is a projective as an \( R \)-module, there exists an \( R \)-homomorphism \( g : N \to R \) such that \( \pi g = fp \). Since \( M \) is a totally fully invariant right ideal of \( R \), then by Proposition 2.4, \( g(N) \subseteq M \). Thus

\[
f(N/J(R)) = fp(N) = \pi g(N) \subseteq \pi(M) \subseteq M/J(R).
\]

Therefore by Proposition 2.4, \( M/J(R) \) is a totally fully invariant right ideal of \( R/J(R) \). □

**Theorem 3.14.** Let \( R \) be a right strongly quasi-duo ring and \( I \) be a proper right ideal of \( R \). Then \( \text{Hom}_R(I, R) \) has no epimorphism and \( \text{Hom}_R(R, I) \) has no monomorphism element.

**Proof.** Let \( I \) be a proper right ideal of \( R \), \( M \) be a maximal right ideal of \( R \) containing \( I \) and \( f \in \text{Hom}_R(I, R) \) be an epimorphism. For each \( b \in R \) there exists \( a \in I \) such that \( f(a) = b \). Then by Proposition 4, \( b \in f(I) \subseteq M \) and hence \( R \subseteq M \). It is a contradiction. Again, Let \( I \) be a proper right ideal of \( R \), \( M \) be a maximal right ideal of \( R \) containing
I and \( f \in \text{Hom}_R(R, I) \) be a monomorphism. Since \( f \) is monic, then for each \( a \in R \), \( \text{ann}_r(a) = \text{ann}_r(f(a)) \). Since \( M \) is a totally fully invariant right ideal of \( R \) and \( f(a) \in M \), then \( \text{ann}_r(a) = \text{ann}_r(f(a)) \) implies that \( a \in M \). Hence \( R \subseteq M \). It is a contradiction. \( \square \)

A ring \( R \) is called directly finite if \( R \) is not isomorphic to a proper summand of itself. By [7, Proposition 1.25], \( R \) is directly finite if and only if for each \( a, b \in R \), \( ab = 1 \) implies that \( ba = 1 \).

**Corollary 3.15.** If \( R \) a right strongly quasi-duo ring, then \( R \) is not isomorphic to any proper right ideal of itself. In particular, \( R \) is a directly finite ring.

**Proof.** It is clear by Theorem 3.14. \( \square \)

**Corollary 3.16.** Right strongly quasi-duo rings are co-hopfian.

**Proof.** Let \( f \in \text{End}(R_R) \) be a monomorphism. Then \( R \cong \text{Im}(f) \). Therefore by Corollary 3.15, \( R = \text{Im}(f) \). \( \square \)

**Proposition 3.17.** If \( R \) is a right strongly quasi-duo ring, then nonequal maximal right ideals of \( R \) are not isomorphic.

**Proof.** Let \( M_1 \) and \( M_2 \) be two maximal right ideals of \( R \) which are isomorphic. Then there exists an \( R \)-isomorphism \( f \in \text{Hom}_R(M_1, M_2) \). For each \( a \in M_1 \), \( \text{ann}_r(a) = \text{ann}_r(f(a)) \). Since \( M_2 \) is a totally fully invariant right ideal of \( R \) and \( f(a) \in M_2 \), then \( \text{ann}_r(f(a)) \subseteq \text{ann}_r(a) \) implies that \( a \in M_2 \). Therefore \( M_1 \subseteq M_2 \) and hence \( M_1 = M_2 \) \( \square \)

**References**


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