Journal of Mathematical Extension
Vol. 5, No. 2 (2), (2011), 123-139

Homotopy Analysis and Padé Methods for Solving Two Nonlinear Equations

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Abstract. In this paper, we are giving analytic approximate solutions to nonlinear PDEs using the Homotopy Analysis Method (HAM) and Homotopy Padé Method(HPadéM). The HAM contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence regions of solution series. It is illustrated that HPadéM accelerates the convergence of the related series. The results reveal these methods are remarkably effective.

AMS Subject Classification: 65M99; 65N99.
Keywords and Phrases: Drinfeld-Sokolov system; Drinfeld-Sokolov-Wilson equation; homotopy analysis method; homotopy padé method.

1. Introduction

Most phenomena in real world are described through nonlinear system equations and these type of equations have attracted lots of attention

Received January 2011; Final Revised October 2011
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among scientists. Large class of nonlinear equations do not have a precise analytic solution, so numerical and analytical methods have largely been used to handle these system equations. These methods include the Homotopy perturbation method [1], Luapanov's artificial small parameter method, δ-expansion method, the tanh-coth method ([2]), $\frac{G'}{G}$-expansion method ([3]), Adomian decomposition method and variational iterative method and so on [4,5].
The Drinfeld-Sokolov equation which formulated as follows

$$\begin{aligned} &u_t + (v^2)_x = 0, \\ &v_t - av_{xxx} + 3bu_xv + 3ku_xv = 0, \end{aligned} \tag{1}$$

where a, b and k are constants, was introduced by Drinfeld and Sokolov as an example of a system of nonlinear equations possessing Lax pairs of a special form [6,7].
The nonlinear Drinfeld Sokolov Wilson Equation(DSWE) is as follows

$$\begin{aligned} &u_t + pvv_t = 0, \\ &v_t + ruv_x + su_xv + qu_{xxx} = 0. \end{aligned} \tag{2}$$

Recently, this equation has been studied by several authors. In [8] and [9] are obtained some exact solutions for DSWE (2) by using a direct algebra methods.
In this paper, we extend the HAM and HPad$\acute{e}$M for approximating the solutions of (1) and (2). HAM firstly was developed by S. J. Liao in 1992. This method can be applied to different type of problems in many engineering and physical applications ([10-15]). HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and rate of the solution series. This method properly overcomes restrictions of perturbation techniques because it does not need any small or large parameters to be contained in the problem. Moreover some methods such as Adomion Decomposition Method and the Homotopy Perturbation Method are special cases of HAM ([16-18]).
There exist some techniques to accelerate the convergence of a given series. Among them, the HPad$\acute{e}$M is widely applied. It is illustrated this technique accelerate the convergence rate of solution series.

2. Homotopy Analysis Method

For convenience of the readers, we will first present a brief description of the standard HAM. For more details the reader can refer to [19-21]. To achieve our goal, let us assume the nonlinear system of differential equations be in the form

$$N_j[u_1(x,t), u_2(x,t), ..., u_m(x,t)] = 0, \quad j = 1...n, \tag{3}$$

where N_j are nonlinear operators, t is an independent variable, $u_i(t)$ are unknown functions. By means of generalizing the traditional homotopy method, Liao construct the zeroth-order deformation equation as follows

$$\begin{aligned}&(1-q)L_j[\phi_i(x,t,q) - u_{i0}(x,t)]\\ &= qhH(t)N_j[\phi_1(x,t,q), \phi_2(x,t,q), ..., \phi_m(x,t,q)],\\ &\quad i = 1...m; \qquad\qquad j = 1, ...n,\end{aligned} \tag{4}$$

where $q \in [0,1]$ is an embedding parameter, L_j are linear operators, $u_{i0}(x,t)$ are initial guesses of $u_i(x,t)$, $\phi_i(x,t;q)$ are unknown functions, h and $H(x,t)$ are auxiliary parameter and auxiliary function respectively. It is important to note that, one has great freedom to choose auxiliary objects such as h and L_j in HAM; This freedom plays an important role in establishing the keystone of validity and flexibility of HAM as shown in this paper. Obviously, when $q = 0$ and $q = 1$, both

$$\phi_i(x,t,0) = u_{i0}(x,t) \quad \text{and} \quad \phi_i(x,t,1) = u_i(x,t), \quad i = 1...m, \tag{5}$$

hold. Thus as q increases from 0 to 1, the solutions of $\phi_i(x,t;q)$ change from the initial guesses $u_{i0}(x,t)$ to the solutions $u_i(x,t)$. Expanding $\phi_i(x,t;q)$ in Taylor series with respect to q, one has

$$\phi_i(x,t,q) = u_{i0}(x,t) + \sum_{k=1}^{+\infty} u_{ik}(x,t)q^k, \quad i = 1...m, \tag{6}$$

where

$$u_{ik}(x,t) \;\; = \;\; \frac{1}{k!}\frac{\partial^k \phi_i(x,t,q)}{\partial q^k}\Big|_{q=0}, \quad i = 1...m. \tag{7}$$

If the auxiliary linear operators, the initial guesses, the auxiliary parameter h, and the auxiliary function are so properly chosen, then the series (6) converges at $q = 1$, then one has

$$\phi_i(x,t,1) = u_{i0}(x,t) + \sum_{k=1}^{+\infty} u_{ik}(x,t), \quad i = 1...m, \tag{8}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao. Define the vectors

$$\vec{u}_{in}(t) = \{u_{i0}(x,t), u_{i1}(x,t), ..., u_{in}(x,t)\}, \quad i = 1...m. \tag{9}$$

Differentiating (4), k times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $k!$, we have the so-called kth-order deformation equation

$$L_j[u_{ik}(x,t) - \chi_k u_{ik-1}(x,t)] = hR_{jk}(\vec{u}_{ik-1}(x,t)),\ i = 1 \cdots m;\ j = 1 \cdots n, \tag{10}$$

subject to the initial conditions

$$L_j(0) = 0,$$

where

$$R_{jk}(\vec{u}_{ik-1}(x,t)) = \frac{1}{(k-1)!} \frac{\partial^{k-1} N_j[\phi_1(x,t,q), \phi_2(x,t,q), \cdots \phi_m(x,t,q)]}{\partial q^{k-1}}\Big|_{q=0}, \tag{11}$$

and

$$\chi_{\mathrm{m}} = \begin{cases} 0 & m \leqslant 1, \\ 1 & m > 1. \end{cases} \tag{12}$$

It should be emphasized that $u_{ik}(x,t)$ is governed by the linear equations (10) and (11) with the linear boundary conditions that come from the original problem. These equations can be easily solved by symbolic computation softwares such as Maple and Mathematica.

3. Homotopy Padé Method

Traditionally the $[m, n]$ Padé for $u(x, t)$ is in the form

$$\frac{\sum_{k=0}^{m} F_k(x)t^k}{1+\sum_{k=1}^{n} F_{m+1+k}(x)t^k},$$

or

$$\frac{\sum_{k=0}^{m} G_k(t)x^k}{1+\sum_{k=1}^{n} G_{k+m+1}(t)x^k},$$

where $F_k(r)$ and $G_k(t)$ are functions.

In Homotopy Padé approximation, we employ the traditional Padé technique to the series (6) for the embedding parameter q to gain the $[m, n]$ Padé approximation in the form of

$$\frac{\sum_{k=0}^{m} w_k(x,t)q^k}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)q^k}, \tag{13}$$

where $w_k(t, x)$ is a function and for $i = 0, 1, \cdots, m, m+2, \cdots, m+n+1$, $w_i(x, t)$ is determined by product of the denominator of the above expression in the $\sum_{i=0}^{m+n} u_i(x,t)q^i$ and equating the powers of q^i, $i = 0, 1, \cdots, m+n$. Thus we have $m+n+1$ equations and $m+n+1$ unknowns $w_i(x, t)$, $i = 0, 1, \cdots, m, m+2, \cdots, m+n+1$. By setting $q = 1$ in (13) the so-called $[m, n]$ Homotopy Padé approximation in the following form is yield.

$$\frac{\sum_{k=0}^{m} w_k(x,t)}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)}. \tag{14}$$

It is found that all of the $[m, n]$ homotopy- Pade approximants do not depend upon the auxiliary parameter h. Thus, even if we choose a bad value of h such that the corresponding solution series diverges, we can still employ the homotopy-Padé technique to get a convergent result. However, up to now, there is not a mathematical proof about it in general cases. All of these illustrate that the so-called homotopy-Pade technique can greatly enlarge the convergence region and rate of the solution series given by the homotopy analysis method. The details of the homotopy-Pade technique can be found in [21].

4. Applications

In this section we apply HAM and HPadéM to solve the aforementioned equations. In all cases, we assume that the initial guesses are $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$ i.e. the initial conditions, and use the auxiliary linear operator $L_j = \frac{\partial}{\partial t}$ and the auxiliary function $H(x,t) = 1$. We give approximations of compute error terms to show the efficiency of HAM and HPadéM.
Solutions in Adomion decomposition Method (ADM), variational iteration method and HAM expressed in the series form respect to initial point, then similar to Taylor expansion method accuracy of approximation in these methods decrease by away from it.

Example 4.1. Let us consider the Drinfeld-Sokolov system (1) by assuming $a = b = 1$ and with exact solution

$$\begin{aligned} v(x,t) &= \csc(x-t), \\ u(x,t) &= \csc^2(x-t). \end{aligned} \tag{15}$$

Employing HAM with mentioned parameters in Section 2, we have the following zero-order deformation equations

$$\begin{aligned} (1-q)L_i[\phi_{1t} - u(x,0)] &= qh[\phi_{1t} + \phi_{2x}^2], \\ (1-q)L_i[\phi_{2t} - v(x,0)] &= qh[\phi_{2t} - \phi_{2xxx} + 3\phi_{1x}\phi_2]. \end{aligned} \tag{16}$$

Of course some authors used various values of h_1 and h_2 in relation (16) and choosed them so that the error of approximation tends to zero [20,22]. This highlights that we have freedom in choose of h_1 and h_2.
Subsequently solving the Nth order deformation equations one has

$$\begin{aligned} u_0(x,t) &= \csc^2(x), \\ v_0(x,t) &= \csc(x), \end{aligned}$$

$$\begin{aligned} u_1(x,t) &= \frac{-2h\cos(x)t}{\sin^3(x)}, \\ v_1(x,t) &= \frac{-h\cos(x)t}{\sin^2(x)}, \end{aligned}$$

$$u_2(x,t) = \frac{ht(-2\cos(x)\sin(x) + 2h\cos^2(x)t + ht - 2h\cos(x)\sin(x)}{\sin^4(x)},$$

$$v_2(x,t) = \frac{1}{2}\frac{ht(-2\cos(x)\sin(x) + h\cos^2(x)t + ht - 2h\cos(x)\sin(x)}{\sin^3(x)},$$

and so.
We use an 9-term approximation and set

$$app8{=}u_0 + u_1 + u_2 + ... + u_8, \quad \text{and} \quad app8{=}v_0 + v_1 + v_2 + ... + v_8.$$

We declare the results for 8th order HAM approximations in Table 1. The influence of h on the convergence of the solution series are given in Figure 1. This figure was obtained by using 8th order HAM approximation for various values of x and t. Moreover the absolute error for $u(x,t)$ and $v(x,t)$ is drawn in Figures 2-5 in 3 dimension for both example 1 and 2. It is easy to see that in order to have a good approximation, h has to be chosen in $-1.8 < h < 0.1$. This means that for these values of h the series (8) converges to the exact solution (15). In Table 2, the absolute error of approximation results is given by $[4,4]$ HPad$\acute{e}$M.

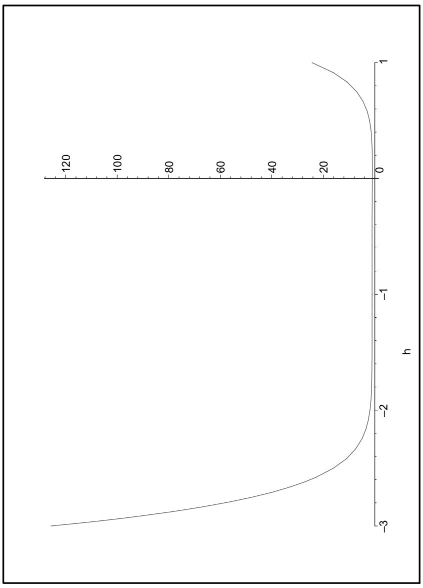

Figure 1: The h curve of Example 4.1 with 8th order HAM

figure[2,3]

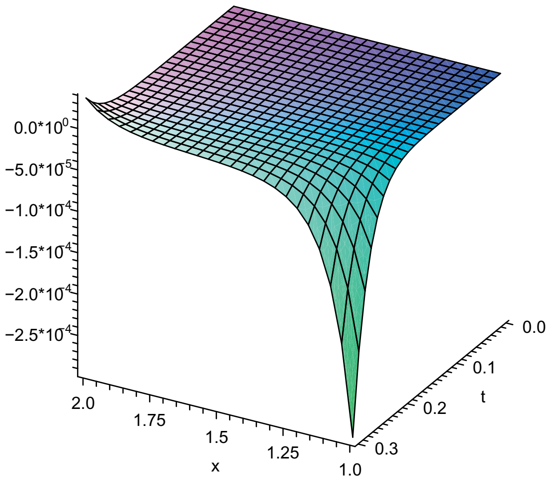

Figure 2: Absolute error curve of Example 4.1 with 8th order HAM of $u(x,t)$ for $1 \le x \le 2$, $0 \le x \le .3$

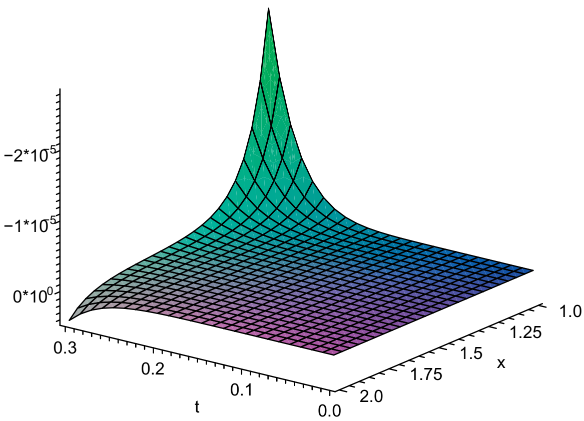

Figure 3: Absolute error curve of Example 4.1 with 8th order HAM of $v(x,t)$ for $1 \le x \le 2$, $0 \le x \le .3$

figure[4,5]

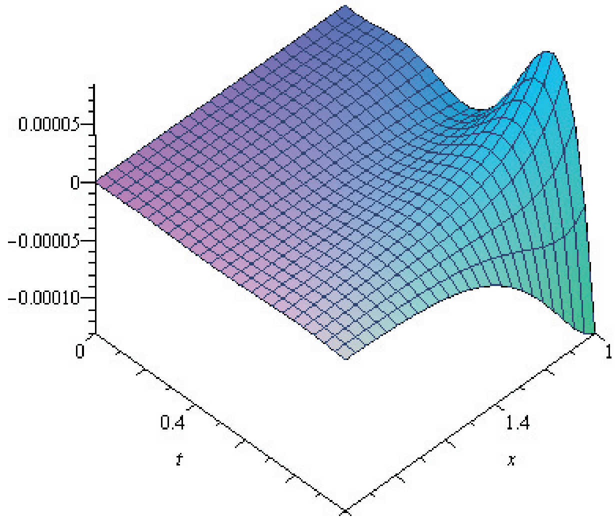

Figure 4: Absolute error curve of Example 4.2 with 8th order HAM of $v(x,t)$ for $1 \leq x \leq 2$, $0 \leq x \leq 1$

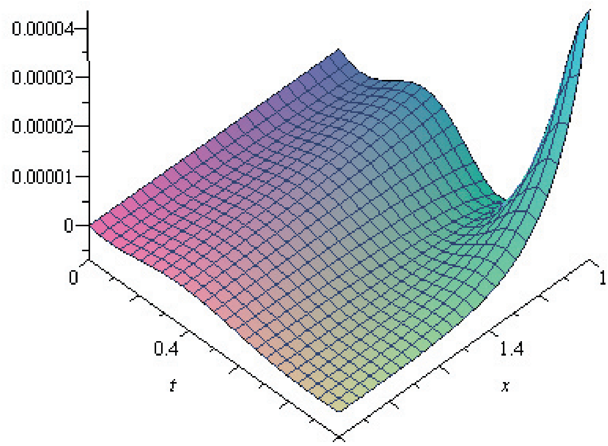

Figure 5: Absolute error curve of Example 4.2 with 8th order HAM of $v(x,t)$ for $1 \leq x \leq 2$, $0 \leq x \leq 1$

The convergence region for $v(x,t)$ is exactly the one for $U(x,t)$. Then we can take in to account one of them and choose suitable value of h from the above region.
In tables 1 and 2, we left the value of $u(1,1)$ blank, because it tends to infinity.

Table 1. Absolute error of Example 4.1 for 8th order HAM with $h = -1.1$

x	t=0.1	t=0.2	t=0.8	t=1
1	(2.95E-8,7.30E-9)	(1.72E-7,1.27E-7)	(8.23E0, 5.47E-1)	–
1.3	(3.50E-8,1.40E-8)	(9.81E-7,2.86E-7)	(1.41E-1,1.30E-2)	(2.68E0,2.35E-1)
1.5	(6.80E-8,2.10E-8)	(2.34E-6,5.93E-7)	(6.97E-3,2.94E-3)	(1.56E-1,5.99E-3)
1.8	(2.57E-7,6.20E-8)	(1.16E-5,2.27E-6)	(1.87E-1,2.94E-2)	(1.03E0, 1.60E-1)
2	(7.92E-7,1.53E-7)	(4.38E-5,7.03E-6)	(9.06E-1,1.19E-1)	(5.14E0, 6.68E-1)

Table 2. Absolute error of Example 4.1 for [4,4] HPadéM

x	t=0.1	t=0.2	t=0.8	t=1
1	(3.47E-12, 3.88E-15)	(2.50E -9, 2.39E-12)	(8.93E-2, 8.28E-6)	
1.3	(7.71E-13, 1.26E-15)	(4.79E-10, 7.30E-13)	(1.40E-3, 7.49E-7)	(6.65E-2, 1.50E-5)
1.5	(1.69E-13, 3.21E-16)	(1.05E-10, 1.93E-13)	(1.88E-4, 1.79E-7)	(4.63E-3, 2.65E-6)
1.8	(4.72E-13,8.74E-16)	(2.23E-10, 4.20E-13)	(7.28E-5, 1.07E-7)	(8.05E-4, 9.31E-7)
2	(1.18E-12,1.89E-15)	(5.30E-10, 8.94E-13)	(1.26E-4, 2.14E-7)	(1.22E-3, 1.81E-6)

Example 4.2. Let us consider again the Drinfeld-Sokolov system (1) by assuming $a = b = 1$ and with exact hyperbolic solution

$$\begin{aligned} v(x,t) &= -\operatorname{sech}(x-t), \\ u(x,t) &= -\operatorname{sech}^2(x-t). \end{aligned} \tag{17}$$

Similar to calculous of example 4.1, one has

$$\begin{aligned} u_0(x,t) &= -\operatorname{sech}^2(x), \\ v_0(x,t) &= -\operatorname{sech}(x), \end{aligned}$$

$$u_1(x,t) = \frac{-2h\sinh(x)t}{\cosh^3(x)},$$

$$v_1(x,t) = \frac{-h\sinh(x)t}{\cosh^2(x)},$$

$$u_2(x,t) = \frac{-ht(2\sinh(x)\cosh(x) + 2ht\cosh^2(x)}{\cosh^4(x)}$$

$$-\frac{3ht + 2h\cosh(x)\sinh(x))}{\cosh^4(x)},$$

$$v_2(x,t) = \frac{-1}{2}\frac{ht(2\sinh(x)\cosh(x) + ht\cosh^2(x)}{\cosh^3(x)}$$

$$-\frac{2ht + 2h\cosh(x)\sinh(x))}{\cosh^3(x)},$$

and so on.

To investigate the suitable h for the convergence of the solution series, we plot the so called h curve of $u(x,t)$ and $v(x,t)$ for various values of x and t. These plots of this problem are not also given here for space consuming. However it is observed that the series of $u(x,t)$ and $v(x,t)$ is convergent when $-1.5 < h < -0.5$. The absolute errors of 8th order approximation solutions by HAM and $[4,4]$ HPad*é*M are reported in some points in Table 3 and Table 4 respectively.

Table 3: Absolute error of Example 4.2 for 8th order HAM

x	t=0.1	t=0.2	t=0.8	t=1
1	(1.32E -8, 2.35E-8)	(1.55E-7,6.94E-8)	(2.24E-3, 1.11E-4)	(1.28E-2, 1.59E-3)
1.3	(2.40E -9, 1.44E-8)	(2.07E-8,2.29E-8)	(8.55E-4, 1.87E-4)	(3.49E-3, 1.14E-3)
1.5	(8.00E-10, 1.09E-8)	(6.70E-9,1.11E-8)	(2.50E-4, 1.06E-4)	(5.44E-4, 5.53E-4)
1.8	(9.00E-11, 7.50E-9)	(2.69E-9,4.40E-9)	(1.69E-5, 3.06E-5)	(4.12E-4, 1.14E-4)
2	(3.27E-11, 5.94E-9)	(1.92E-9,2.42E-9)	(3.45E-5, 1.08E-5)	(3.39E-4, 2.12E-5)

. Table 4: Absolute error of Example 4.2 for [4,4] HPadéM

x	t=0.1	t=0.2	t=0.8	t=1
1	(9.11E-10, 1.71E-11)	(8.75E-8, 1.92E-9)	(2.77E-4, 1.38E-5)	(7.95E-4, 4.99E-5)
1.3	(5.75E-10, 1.02E-11)	(6.85E-8, 1.13E-9)	(1.82E-3, 7.71E-6)	(1.29E-2, 2.75E-5)
1.5	(1.04E-10, 7.91E-12)	(1.11E-8,8.68E-10)	(6.56E-5, 5.73E-6)	(2.27E-4, 2.04E-5)
1.8	(2.21E-11, 6.24E-12)	(2.33E-9,6.77E-10)	(1.23E-5, 4.26E-6)	(4.11E-5, 1.50E-5)
2	(1.06E-11, 6.33E-12)	(1.11E-9,6.76E-10)	(5.95E-6, 4.01E-6)	(1.99E-5, 1.39E-5)

Example 4.3. Now we consider the DSWE (2) by letting $p = s = -1$, $r = 3$, $q = 1$ and with the exact solutions

$$\begin{aligned} u(x,t) &= 2\sqrt{3}\tanh(x-t), \\ v(x,t) &= -5 + 6\mathrm{sech}(x-t). \end{aligned} \tag{18}$$

Employing HAM with $L = \frac{\partial}{\partial t}$, $H(x,t) = 1$, $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$, we have the following zero-order deformation equation

$$\begin{aligned} (1-q)L_i[\phi_{1t} - u_{0t}(t)] &= qh[\phi_{1t} - \phi_2\phi_{2t}], \qquad (19) \\ (1-q)L_i[\phi_{2t} - v_{0t}(t)] &= qh[\phi_{2t} + 3\phi_{1x}\phi_{2x} - \phi_{1x}\phi_2 + \phi_{1xxx}]. \end{aligned}$$

Subsequently solving the Nth order deformation equations one has

$$\begin{aligned} u_0(x,t) &= 2\sqrt{3}\tanh(x), \\ v_0(x,t) &= -5 + 6\mathrm{sech}^2(x), \end{aligned}$$

$$\begin{aligned} u_1(x,t) &= \frac{-12h\sinh(x)t}{\cosh^3(x)}, \\ v_1(x,t) &= \frac{2h\sqrt{3}t}{\cosh^2(x)}, \end{aligned}$$

$$u_2(x,t) = \frac{-2ht(6\cosh^2(x)\sinh(x) + 12h\sqrt{3}t\cosh(x)^2\sinh(x)}{\cosh^5(x)}$$

$$-\frac{24\sinh(x)h\sqrt{3}t - h\sqrt{3}\cosh^3(x))}{\cosh^5(x)},$$

$$v_2(x,t) = \frac{-2ht(-\sqrt{3}\cosh^4(x) + 60ht\cosh^4(x) - 30ht\cosh^2(x)}{\cosh^6(x)}$$

$$-\frac{54ht + 6h\cosh^3(x)\sinh(x))}{\cosh^6(x)},$$

and so.

To investigate the suitable h for the convergence of the solution series, we plot the so called h curve of $u(x,t)$ and $v(x,t)$ for various values of x and t. These plots of this problem are not given here for space consuming. However it is observed that the series of $u(x,t)$ and $v(x,t)$ is convergent when $-1.5 < h < -0.5$. The absolute errors of 8th order approximation solutions by HAM and $[4,4]$ HPad$\acute{e}$M are reported in some points in Table 5. and Table 6. respectively.

Table 5. Absolute error of Example 4.3 for 8th order HAM

x	t=0.1	t=0.2	t=0.8	t=1
1	(3.16E-8,9.17E-8)	(3.31E-7,9.59E-7)	(1.11E-2,3.75E-2)	(6.07E-2,2.31E-1)
1.3	(1.90E-8,6.40E-8)	(1.47E-7,8.23E-7)	(3.34E-3,3.53E-2)	(1.60E-2,1.97E-1)
1.5	(1.03E-8,4.20E-8)	(7.79E-8,4.30E-7)	(5.52E-4,1.42E-2)	(7.24E-4,7.52E-2)
1.8	(8.66E-9,2.30E-8)	(3.11E-8,1.51E-7)	(4.43E-4,5.18E-4)	(3.86E-3,2.91E-3)
2.3	(3.46E-9,7.00E-9)	(6.92E-9,3.30E-8)	(1.76E-4,9.20E-4)	(1.24E-3,7.06E-3)
2.7	(0.00000,4.00E-9)	(3.46E-9,1.40E-8)	(4.61E-5,2.86E-4)	(2.84E-4,2.00E-3)
3	(4.9E-10,2.03E-9)	(1.66E-9,5.56E-9)	(1.53E-5,9.94E-5)	(7.58E-5,6.32E-4)

Table 6. Absolute error of Example 4.3 for [4,4] HPad$\acute{e}$M

x	t=0.1	t=0.2	t=0.8	t=1
1	(1.07E-10, 9.79E-9)	(1.58E-8, 1.68E-6)	(4.89E-4,6.70E-1)	(2.56E-3,8.60E-1)
1.3	(6.71E-11, 4.11E-9)	(1.01E-8, 5.78E-7)	(4.00E-4,1.44E-2)	(2.34E-3,6.94E-2)
1.5	(1.04E-10,7.91E-12)	(1.11E-8,8.60E-10)	(6.56E-5,5.73E-6)	(2.27E-4,2.04E-5)
1.8	(2.75E-11,1.97E-10)	(4.26E-9, 3.08E-8)	(2.13E-4,1.53E-3)	(1.43E-3,9.91E-3)
2.3	(1.05E-11,4.11E-11)	(1.64E-9, 6.42E-9)	(9.19E-5,3.51E-4)	(6.60E-4,2.46E-3)
2.7	(4.80E-12,1.69E-11)	(7.53E-10,2.65E-9)	(4.35E-5,1.50E-4)	(3.20E-4,1.09E-3)
3	(2.65E-12,9.21E-12)	(4.15E-10,1.44E-9)	(2.43E-5,8.35E-5)	(1.81E-4,6.14E-4)

5. Conclusion

In this paper, we approximate the solutions of the Drinfeld-Sokolov equation and DSWE by the HAM and HPad$\acute{e}$M. The convergence region for our approximation, are determined by the parameter h, which provides us a great freedom to choose convenient value for it. It is illustrated efficiency and accuracy of proposed methods by implementing on the mentioned equations. It is shown the HPad$\acute{e}$M accelerate the convergence of the related series.

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