Approximate Additive Functional Equations in Closed Convex Cone

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Abstract. In this paper, we introduce the following positive-additive functional equation in $C^*$-algebras

\[
f\left(x + 4 \sqrt[4]{x^2y} + 6 \sqrt[6]{xy} + 4 \sqrt[4]{xy^3} + y\right) = \\
f(x) + 4f(x) \frac{4}{1} \sqrt[4]{f(y)} + 6 \sqrt[6]{f(x)f(y)} + 4f(y) \frac{4}{1} \sqrt[4]{f(x)} + f(y).
\]

Using the fixed point method, we prove the stability of the positive-additive functional equation in $C^*$-algebras. Moreover, we prove the Hyers-Ulam stability of the above functional equation in $C^*$-algebras by the direct method of Hyers-Ulam.

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1. Introduction

The stability problem of functional equations was originated from a question of Ulam ([43]) concerning the stability of group homomorphisms. Hyers ([24]) gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki ([1]) for additive mappings and by Th.M. Rassias ([39]) for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. (T. M. Rassias) Let $f$ be an approximately additive mapping from a normed vector space $E$ into a Banach space $E'$, i.e., $f$
satisfies the inequality
\[ \frac{|f(x + y) - f(x) - f(y)|}{\|x\|^r + \|y\|^r} \leq \epsilon \]
for all \( x, y \in E - \{0\} \), where \( \epsilon \) and \( r \) are constants with \( \epsilon > 0 \) and \( 0 \leq r < 1 \). Then the mapping \( L : E \rightarrow E' \) defined by \( L(x) := \lim_{n \to \infty} 2^{-n} f(2^n x) \) is the unique additive mapping which satisfies,
\[ \frac{|f(x) - L(x)|}{\|x\|^r} \leq \frac{2\epsilon}{2 - 2^r}, \]
for all \( x \in E - \{0\} \).

The paper of Th.M. Rassias ([39]) has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Gavruta ([20]) by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias’ approach. J.M. Rassias [36]-[38] followed the innovative approach of the Th.M. Rassias’ theorem [39] in which he replaced the factor \( \|x\|^p + \|y\|^p \) by \( \|x\|^p - \|y\|^q \) for \( p, q \in \mathbb{R} \) with \( p + q \neq 1 \). The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]-[15],[17]-[42]).

**Definition 1.2.** [16] Let \( A \) be a \( C^* \)-algebra and \( x \in A \) a self-adjoint element, i.e., \( x^* = x \). Then \( x \) is said to be positive if it is of the form \( yy^* \) for some \( y \in A \). The set of positive elements of \( A \) is denoted by \( A^+ \).

Note that \( A^+ \) is a closed convex cone (see [16]). It is well-known that for a positive element \( x \) and a positive integer \( n \) there exists a unique positive element \( y \in A^+ \) such that \( x = y^n \). We denote \( y \) by \( x^{\frac{1}{n}} \) (see [16]).

In this paper, we introduce the following functional equation
\[ f \left( x + 4\sqrt[4]{x^3 y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y \right) = f(x) + 4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)} + 6\sqrt{f(x)}f(y) + 4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)} + f(y) \quad (1) \]
in the set of for all \( x, y \in A^+ \), which is called a positive-additive functional equation. Each solution of the positive-additive functional equation is called a positive-additive mapping.

Note that the function \( f(x) = cx, \quad c \geq 0 \), in the set of non-negative real numbers is a solution of the functional equation (1).

Let \( X \) be a set. A function \( d : X \times X \to [0, \infty] \) is called a generalized metric on \( X \) if \( d \) satisfies

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).

We recall a fundamental result in fixed point theory.

**Theorem 1.3.** Let \( (X, d) \) be a complete generalized metric space and let \( J : X \to X \) be a strictly contractive mapping with Lipschitz constant \( L < 1 \). Then for each given element \( x \in X \), either

\[
d(J^n x, J^{n+1} x) = \infty
\]

for all nonnegative integers \( n \) or there exists a positive integer \( n_0 \) such that

1. \( d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0 \);
2. the sequence \( \{J^n x\} \) converges to a fixed point \( y^* \) of \( J \);
3. \( y^* \) is the unique fixed point of \( J \) in the set \( Y = \{ y \in X \mid d(J^{n_0} x, y) < \infty \} \);
4. \( d(y, y^*) \leq \frac{1}{1-L} d(y, Jy) \) for all \( y \in Y \).

In 1991, Baker ([10]) used the Banach fixed point theorem to give generalized Hyers-Ulam stability results for a nonlinear functional equation. In 2003, Radu ([35]) applied the fixed point alternative theorem to prove the generalized Hyers-Ulam stability. Miheţ ([29]) applied the Luxemburg-Jung fixed point theorem in generalized metric spaces to study the generalized Hyers-Ulam stability for two functional equations in a single variable and L. Găvruta ([19]) used the Matkowski’s fixed point theorem to obtain a new general result concerning the generalized Hyers-Ulam stability of a functional equation in a single variable. In 1996, G. Isac and Th.M. Rassias ([26]) were the first to provide appli-
cations of stability theory of functional equations for the proof of new fixed point theorems with applications.

This paper is organized as follows: In Section 2, using the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1) in $C^*$-algebras. In Section 3, using the direct method, we prove the Hyers-Ulam stability of the functional equation (1) in $C^*$-algebras.

Throughout this paper, let $A^+$ and $B^+$ be the sets of positive elements in $C^*$-algebras $A$ and $B$, respectively.

2. Stability of Eq. (1): Fixed Point Approach

In this section, we investigate the positive-additive functional equation (1) in $C^*$-algebras.

Lemma 2.1. Let $T : A^+ \to B^+$ be a positive-additive mapping satisfying (1). Then $T$ satisfies $T(16^n x) = 16^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$.

Proof. Putting $x = y$ in (1.1), we obtain $T(16x) = 16T(x)$ for all $x \in A^+$. By induction on $n$, one can show that $T(16^n x) = 16^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in $C^*$-algebras. Note that the fundamental ideas in the proofs of the main results in this section are contained in [12, 13].

Theorem 2.2. Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\frac{16}{L} \varphi \left( \frac{x}{16}, \frac{y}{16} \right) \leq \varphi(x, y)$$ (2)

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$\left\| f \left( x + 4\sqrt[3]{x^3 y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y \right) - f(x) - 4f(x)^{\frac{3}{2}} \sqrt[4]{f(y)} \right\| + \left| 6\sqrt{f(x)f(y)} - 4f(y)^{\frac{3}{2}} \sqrt[4]{f(x)} \right| - f(y) \right\| \leq \varphi(x, y)$$ (3)
for all \( x, y \in A^+ \). Then there exists a unique positive-additive mapping \( A : A^+ \to A^+ \) satisfying (1) and

\[
\| f(x) - A(x) \| \leq \frac{L \varphi(x, x)}{16 - 16L}
\]

for all \( x \in A^+ \).

**Proof.** Letting \( y = x \) in (3), we get

\[
\| f(16x) - 16f(x) \| \leq \varphi(x, x)
\]

for all \( x \in A^+ \). Consider the set

\[
X := \{ g : A^+ \to B^+ \}
\]

and introduce the generalized metric on \( X \):

\[
d(g, h) = \inf\{ \mu \in (0, +\infty) : \| g(x) - h(x) \| \leq \mu \varphi(x, x), \ \forall x \in A^+ \},
\]

where, as usual, \( \inf \phi = +\infty \). It is easy to show that \( (X, d) \) is complete (see [30]). Now we consider the linear mapping \( J : X \to X \) such that

\[
Jg(x) := 16g \left( \frac{x}{16} \right)
\]

for all \( x \in A^+ \). Let \( g, h \in X \) be given such that \( d(g, h) = \varepsilon \). Then, \( \| g(x) - h(x) \| \leq \varphi(x, x) \) for all \( x \in A^+ \). Hence

\[
\| Jg(x) - Jh(x) \| = \left\| 16g \left( \frac{x}{16} \right) - 16h \left( \frac{x}{16} \right) \right\| \leq L \varphi(x, x)
\]

for all \( x \in A^+ \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L\varepsilon \). This means that, \( d(Jg, Jh) \leq Ld(g, h) \) for all \( g, h \in X \).

It follows from (5) that

\[
\left\| f(x) - 16f \left( \frac{x}{16} \right) \right\| \leq \frac{L}{16} \varphi(x, x)
\]

for all \( x \in A^+ \). So \( d(f, Jf) \leq \frac{L}{16} \). By Theorem 1.3., there exists a mapping \( A : A^+ \to B^+ \) satisfying the following:
(1) $A$ is a fixed point of $J$, i.e.,
\[ A \left( \frac{x}{16} \right) = \frac{1}{16} A(x) \] (6)
for all $x \in A^+$. The mapping $A$ is a unique fixed point of $J$ in the set $M = \{ g \in X : d(f, g) < \infty \}$. This implies that $A$ is a unique mapping satisfying (6) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - A(x)\| \leq \mu \varphi(x, x)$ for all $x \in A^+$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality
\[ \lim_{n \to \infty} 16^n f \left( \frac{x}{16^n} \right) = A(x) \]
for all $x \in A^+$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality
\[ d(f, A) \leq \frac{L}{16 - 16L} \]
This implies that the inequality (4) holds. By (2) and (3),
\[
\begin{align*}
&\left\| A \left( x + 4 \sqrt[4]{x^3y} + 6 \sqrt[4]{xy^3} + y \right) - A(x) \\
&- 4A(x) \frac{3}{4} \sqrt[4]{A(y)} - 6 \sqrt[4]{A(x)A(y)} - 4A(y) \frac{3}{4} \sqrt[4]{A(x)} - A(y) \right\| \\
&= \lim_{n \to +\infty} \left\| 16^n \left[ f \left( \frac{x}{16^n} + 4 \sqrt[4]{\frac{x^3y}{65536^n}} + 6 \sqrt[4]{\frac{xy^3}{256^n}} + 4 \sqrt[4]{\frac{xy^3}{65536^n}} + \frac{y}{16^n} \right) \\
- f \left( \frac{x}{16^n} \right) - 4f \left( \frac{x}{16^n} \right) \frac{3}{4} \sqrt[4]{f\left( \frac{y}{16^n} \right)} - 6 \sqrt[4]{f\left( \frac{x}{16^n} \right)f\left( \frac{y}{16^n} \right)} \\
- 4f \left( \frac{y}{16^n} \right) \frac{3}{4} \sqrt[4]{f\left( \frac{x}{16^n} \right) - f\left( \frac{y}{16^n} \right)} \right] \right\| \\
&\leq \lim_{n \to +\infty} 16^n \varphi \left( \frac{x}{16^n}, \frac{y}{16^n} \right) \\
&\leq \lim_{n \to +\infty} 16^n \times \frac{L^n}{16^n} \varphi(x, y) \\
&= 0
\end{align*}
\]
for all $x, y \in A^+$. So
\[ A \left( x + 4 \sqrt[4]{x^3y} + 6 \sqrt[4]{xy^3} + 4 \sqrt[4]{xy^3} + y \right) = A(x) + 4A(x) \frac{3}{4} \sqrt[4]{A(y)} \]
for all \( x, y \in A^+ \). Thus the mapping \( A : A^+ \to B^+ \) is positive-additive, as desired. \( \square \)

**Corollary 2.3.** Let \( p > 1 \) and \( \theta_1, \theta_2 \) be non-negative real numbers, and let \( f : A^+ \to B^+ \) be a mapping such that

\[
\| f \left( x + 4\sqrt[4]{x^3y + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y} \right) - f(x) \| \leq \theta_1(\| x \|^p + \| y \|^p) + \theta_2 \cdot \| x \|^{\frac{p}{2}} \cdot \| y \|^{\frac{p}{2}}
\]

for all \( x, y \in A^+ \). Then there exists a unique positive-additive mapping \( A : A^+ \to B^+ \) satisfying (1) and

\[
\| f(x) - A(x) \| \leq \frac{(2\theta_1 + \theta_2)\| x \|^p}{16^p - 16}
\]

for all \( x \in A^+ \).

**Proof.** The proof follows from Theorem 2.2 by taking \( \varphi(x, y) = \theta_1(\| x \|^p + \| y \|^p) + \theta_2 \cdot \| x \|^{\frac{p}{2}} \cdot \| y \|^{\frac{p}{2}} \) for all \( x, y \in A^+ \). Then we can choose \( L = 16^{1-p} \) and we get the desired result. \( \square \)

**Theorem 2.4.** Let \( \varphi : A^+ \times A^+ \to [0, \infty) \) be a function such that there exists an \( \alpha < 1 \) with

\[
\varphi(x, y) \leq 16L \varphi \left( \frac{x}{16}, \frac{y}{16} \right)
\]

for all \( x, y \in A^+ \). Let \( f : A^+ \to B^+ \) be a mapping satisfying (3). Then there exists a unique positive-additive mapping \( A : A^+ \to A^+ \) satisfying (1) and

\[
\| f(x) - A(x) \| \leq \frac{\varphi(x, x)}{16 - 16L}
\]

for all \( x \in A^+ \).
**Proof.** Let \((X, d)\) be the generalized metric space defined in the proof of Theorem 2.2.
Consider the linear mapping \(J : X \to X\) such that
\[
Jg(x) := \frac{1}{16}g(16x)
\]
for all \(x \in A^+\).
It follows from (5) that
\[
\left\| f(x) - \frac{f(16x)}{16} \right\| \leq \frac{1}{16} \varphi(x, x)
\]
for all \(x \in A^+\). So \(d(f, Jf) \leq \frac{1}{16}\).
The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** Let \(0 < p < 1\) and \(\theta_1, \theta_2\) be non-negative real numbers, and let \(f : A^+ \to B^+\) be a mapping satisfying (7). Then there exists a unique positive-additive mapping \(A : A^+ \to B^+\) satisfying (1) and
\[
\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{16 - 16^p} \|x\|^p
\]
for all \(x \in A^+\).

**Proof.** The proof follows from Theorem 2.4 by taking \(\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}\) for all \(x, y \in A^+\). Then we can choose \(L = 16^{p-1}\) and we get the desired result. □

### 3. Stability of Eq. (1): Direct Method

In this section, using the direct method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in \(C^*-\)algebras.

**Theorem 3.1.** Let \(f : A^+ \to B^+\) be a mapping for which there exists a function \(\varphi : A^+ \times A^+ \to [0, \infty)\) satisfying (3) and
\[
\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 16^j \varphi \left( \frac{x}{16^j}, \frac{y}{16^j} \right) < \infty \quad (8)
\]
for all \( x, y \in A^+ \). Then there exists a unique positive-additive mapping \( A : A^+ \to A^+ \) satisfying (1) and

\[
\| f(x) - A(x) \| \leq \frac{1}{16} \tilde{\varphi}(x, x)
\]  

for all \( x \in A^+ \).

**Proof.** It follows from (5) that

\[
\left\| f(x) - 16f\left(\frac{x}{16}\right) \right\| \leq \varphi\left(\frac{x}{16}, \frac{x}{16}\right)
\]

for all \( x \in A^+ \). Hence

\[
\left\| 16^l f\left(\frac{x}{16^l}\right) - 16^k f\left(\frac{x}{16^k}\right) \right\| \leq \frac{1}{16} \sum_{j=l+1}^{k} 16^j \varphi\left(\frac{x}{16^j}, \frac{x}{16^j}\right)
\]

for all nonnegative integers \( k \) and \( l \) with \( k > l \) and all \( x \in A^+ \). It follows from (8) and (10) that the sequence \( \{16^j f\left(\frac{x}{16^j}\right)\} \) is Cauchy for all \( x \in A^+ \). Since \( B^+ \) is complete, the sequence \( \{16^j f\left(\frac{x}{16^j}\right)\} \) converges. So one can define the mapping \( A : A^+ \to B^+ \) by

\[
A(x) := \lim_{j \to \infty} 16^j f\left(\frac{x}{16^j}\right)
\]

for all \( x \in A^+ \). By (3) and (8),

\[
\left\| A \left( x + 4\sqrt[3]{x^3y} + 6\sqrt{xy} + 4\sqrt[3]{xy^3} + y \right) - A(x) - 4f(x) \sqrt[3]{A(y)} - 6\sqrt{A(x)A(y)} - 4A(y) \sqrt[3]{A(x)} - A(y) \right\|
\]

\[
= \lim_{n \to +\infty} \left\| 16^n \left[ f\left(\frac{x}{16^n} + 4\sqrt[3]{\frac{x^3y}{65536^n}} + 6\sqrt{\frac{xy}{256^n}} + 4\sqrt[3]{\frac{xy^3}{65536^n}} + \frac{y}{16^n}\right) \right.ight.
\]

\[
-4f\left(\frac{x}{16^n}\right) - 4f\left(\frac{x}{16^n}\right) \sqrt[3]{f\left(\frac{y}{16^n}\right)} - 6\sqrt{f\left(\frac{x}{16^n}\right)f\left(\frac{y}{16^n}\right)}
\]

\[
-4f\left(\frac{y}{16^n}\right) \sqrt[3]{f\left(\frac{x}{16^n}\right)} - f\left(\frac{y}{16^n}\right) \left. \right\|
\]

\[
\leq \lim_{n \to +\infty} 16^n \varphi\left(\frac{x}{16^n}, \frac{y}{16^n}\right)
\]

\[
= 0
\]
for all $x, y \in A^+$. So

$$
\left\| A \left( x + 4 \sqrt[4]{x^3} y + 6 \sqrt{xy} + 4 \sqrt[4]{xy^3} + y \right) - A(x) - 4A(x)^{\frac{3}{4}} \right\| = 0
$$

for all $x, y \in A^+$. Hence the mapping $A : A^+ \to B^+$ is positive-additive. Moreover, letting $l = 0$ and passing the limit $k \to \infty$ in (10), we get (9). So there exists a positive-additive mapping $A : A^+ \to B^+$ satisfying (1) and (9).

Now, let $B : A^+ \to B^+$ be another positive-additive mapping satisfying (1) and (9). Then we have

$$
\|A(x) - B(x)\| = 16^q \left\| A \left( \frac{x}{16^q} \right) - B \left( \frac{x}{16^q} \right) \right\| \\
\leq 16^q \left\| A \left( \frac{x}{16^q} \right) - f \left( \frac{x}{16^q} \right) \right\| + 16^q \left\| B \left( \frac{x}{16^q} \right) - f \left( \frac{x}{16^q} \right) \right\| \\
\leq 2 \cdot 16^{q-1} \tilde{\varphi} \left( \frac{x}{16^q}, \frac{x}{16^q} \right),
$$

which tends to zero as $q \to \infty$ for all $x \in A^+$. So we can conclude that $A(x) = B(x)$ for all $x \in A^+$. This proves the uniqueness of $A$. □

**Corollary 3.2.** Let $p > 1$ and $\theta_1, \theta_2$ be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying (7). Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying (1) and

$$
\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{16^p - 16} \|x\|^p
$$

for all $x \in A^+$.

**Proof.** Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.1. Then we get the desired result. □

**Theorem 3.3.** Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying (3) such that

$$
\bar{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{\varphi(16^j x, 16^j y)}{16^j} < \infty
$$
for all \( x, y \in A^+ \). Then there exists a unique positive-additive mapping 
\[ A : A^+ \to B^+ \] 
 satisfying (1) and 
\[ \| f(x) - A(x) \| \leq \frac{1}{16} \varphi(x, x) \]
for all \( x \in A^+ \).

**Proof.** It follows from (5) that 
\[ \left\| f(x) - \frac{f(16x)}{16} \right\| \leq \frac{1}{16} \varphi(x, x) \]
for all \( x \in A^+ \). The rest of the proof is similar to the proof of Theorem 3.1. \( \square \)

**Corollary 3.4.** Let \( 0 < p < 1 \) and \( \theta_1, \theta_2 \) be non-negative real numbers, and let \( f : A^+ \to B^+ \) be a mapping satisfying (7). Then there exists a unique positive-additive mapping 
\[ A : A^+ \to B^+ \] 
 satisfying (1) and 
\[ \| f(x) - A(x) \| \leq \frac{2\theta_1 + \theta_2}{16 - 16^p} \| x \|^p \]
for all \( x \in A^+ \).

**Proof.** Define \( \varphi(x, y) = \theta_1(\| x \|^p + \| y \|^p) + \theta_2 \cdot \| x \|^p \cdot \| y \|^p \), and apply Theorem 3.3. Then we get the desired result. \( \square \)

**References**


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