

Approximate Additive Functional Equations in Closed Convex Cone

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Abstract. In this paper, we introduce the following positive-additive functional equation in C^* -algebras

$$f\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) =$$

$$f(x) + 4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)} + 6\sqrt{f(x)f(y)} + 4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)} + f(y).$$

Using the fixed point method, we prove the stability of the positive-additive functional equation in C^* -algebras. Moreover, we prove the Hyers-Ulam stability of the above functional equation in C^* -algebras by the direct method of Hyers-Ulam.

AMS Subject Classification: 46L05; 47H10; 39B52.

Keywords and Phrases: Hyers-Ulam stability, C^* -algebra, fixed point, positive-additive functional equation.

1. Introduction

The stability problem of functional equations was originated from a question of Ulam ([43]) concerning the stability of group homomorphisms. Hyers ([24]) gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki ([1]) for additive mappings and by Th.M. Rassias ([39]) for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. (*T. M. Rassias*) Let f be an approximately additive mapping from a normed vector space E into a Banach space E' , i.e., f

Received November 2010; Final Revised January 2011

satisfies the inequality

$$\frac{|f(x+y) - f(x) - f(y)|}{\|x\|^r + \|y\|^r} \leq \epsilon$$

for all $x, y \in E - \{0\}$, where ϵ and r are constants with $\epsilon > 0$ and $0 \leq r < 1$. Then the mapping $L : E \rightarrow E'$ defined by $L(x) := \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies,

$$\frac{|f(x) - L(x)|}{|x|^r} \leq \frac{2\epsilon}{2 - 2^r},$$

for all $x \in E - \{0\}$.

The paper of Th.M. Rassias ([39]) has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta ([20]) by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [36]-[38] followed the innovative approach of the Th.M. Rassias' theorem [39] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]-[15], [17]-[42]).

Definition 1.2. [16] Let A be a C^* -algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be positive if it is of the form yy^* for some $y \in A$. The set of positive elements of A is denoted by A^+ .

Note that A^+ is a closed convex cone (see [16]). It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [16]).

In this paper, we introduce the following functional equation

$$\begin{aligned} f\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) &= f(x) + 4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)} \\ &+ 6\sqrt{f(x)f(y)} \\ &+ 4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)} + f(y) \end{aligned} \quad (1)$$

in the set of for all $x, y \in A^+$, which is called a *positive-additive functional equation*. Each solution of the positive-additive functional equation is called a *positive-additive mapping*.

Note that the function $f(x) = cx$, $c \geq 0$, in the set of non-negative real numbers is a solution of the functional equation (1).

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.3. *Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
- (2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
- (3) y^* *is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1991, Baker ([10]) used the Banach fixed point theorem to give generalized Hyers-Ulam stability results for a nonlinear functional equation. In 2003, Radu ([35]) applied the fixed point alternative theorem to prove the generalized Hyers-Ulam stability. Mihet ([29]) applied the Luxemburg-Jung fixed point theorem in generalized metric spaces to study the generalized Hyers-Ulam stability for two functional equations in a single variable and L. Găvruta ([19]) used the Matkowski's fixed point theorem to obtain a new general result concerning the generalized Hyers-Ulam stability of a functional equation in a single variable. In 1996, G. Isac and Th.M. Rassias ([26]) were the first to provide appli-

cations of stability theory of functional equations for the proof of new fixed point theorems with applications.

This paper is organized as follows: In Section 2, using the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1) in C^* -algebras. In Section 3, using the direct method, we prove the Hyers-Ulam stability of the functional equation (1) in C^* -algebras.

Throughout this paper, let A^+ and B^+ be the sets of positive elements in C^* -algebras A and B , respectively.

2. Stability of Eq. (1): Fixed Point Approach

In this section, we investigate the positive-additive functional equation (1) in C^* -algebras.

Lemma 2.1. *Let $T : A^+ \rightarrow B^+$ be a positive-additive mapping satisfying (1). Then T satisfies $T(16^n x) = 16^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$.*

Proof. Putting $x = y$ in (1.1), we obtain $T(16x) = 16T(x)$ for all $x \in A^+$. By induction on n , one can show that $T(16^n x) = 16^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$. $\square$

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in C^* -algebras. Note that the fundamental ideas in the proofs of the main results in this section are contained in [12, 13].

Theorem 2.2. *Let $\varphi : A^+ \times A^+ \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\frac{16}{L} \varphi\left(\frac{x}{16}, \frac{y}{16}\right) \leq \varphi(x, y) \quad (2)$$

for all $x, y \in A^+$. Let $f : A^+ \rightarrow B^+$ be a mapping satisfying

$$\begin{aligned} & \left\| f\left(x + 4\sqrt[4]{x^3 y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) - f(x) \right. \\ & \quad \left. - 4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)} - 6\sqrt{f(x)f(y)} - 4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)} - f(y) \right\| \\ & \leq \varphi(x, y) \end{aligned} \quad (3)$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $\mathbf{A} : A^+ \rightarrow A^+$ satisfying (1) and

$$\|f(x) - \mathbf{A}(x)\| \leq \frac{L\varphi(x, x)}{16 - 16L} \quad (4)$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (3), we get

$$\|f(16x) - 16f(x)\| \leq \varphi(x, x) \quad (5)$$

for all $x \in A^+$. Consider the set

$$X := \{g : A^+ \rightarrow B^+\}$$

and introduce the generalized metric on X :

$$d(g, h) = \inf\{\mu \in (0, +\infty) : \|g(x) - h(x)\| \leq \mu\varphi(x, x), \quad \forall x \in A^+\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (X, d) is complete (see [30]). Now we consider the linear mapping $J : X \rightarrow X$ such that

$$Jg(x) := 16g\left(\frac{x}{16}\right)$$

for all $x \in A^+$. Let $g, h \in X$ be given such that $d(g, h) = \varepsilon$. Then, $\|g(x) - h(x)\| \leq \varphi(x, x)$ for all $x \in A^+$. Hence

$$\|Jg(x) - Jh(x)\| = \left\| 16g\left(\frac{x}{16}\right) - 16h\left(\frac{x}{16}\right) \right\| \leq L\varphi(x, x)$$

for all $x \in A^+$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that, $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in X$.

It follows from (5) that

$$\left\| f(x) - 16f\left(\frac{x}{16}\right) \right\| \leq \frac{L}{16}\varphi(x, x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{L}{16}$. By Theorem 1.3., there exists a mapping $A : A^+ \rightarrow B^+$ satisfying the following:

(1) A is a fixed point of J , i.e.,

$$A\left(\frac{x}{16}\right) = \frac{1}{16}A(x) \quad (6)$$

for all $x \in A^+$. The mapping A is a unique fixed point of J in the set $M = \{g \in X : d(f, g) < \infty\}$. This implies that $\mathbf{A}$ is a unique mapping satisfying (6) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - A(x)\| \leq \mu\varphi(x, x)$ for all $x \in A^+$;

(2) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} 16^n f\left(\frac{x}{16^n}\right) = A(x)$$

for all $x \in A^+$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{16 - 16L}.$$

This implies that the inequality (4) holds. By (2) and (3),

$$\begin{aligned} & \left\| A\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) - A(x) \right. \\ & \quad \left. - 4A(x)^{\frac{3}{4}}\sqrt[4]{A(y)} - 6\sqrt{A(x)A(y)} - 4A(y)^{\frac{3}{4}}\sqrt[4]{A(x)} - A(y) \right\| \\ &= \lim_{n \rightarrow +\infty} \left\| 16^n \left[f\left(\frac{x}{16^n} + 4\sqrt[4]{\frac{x^3y}{65536^n}} + 6\sqrt{\frac{xy}{256^n}} + 4\sqrt[4]{\frac{xy^3}{65536^n}} + \frac{y}{16^n}\right) \right. \right. \\ & \quad \left. - f\left(\frac{x}{16^n}\right) - 4f\left(\frac{x}{16^n}\right)^{\frac{3}{4}}\sqrt[4]{f\left(\frac{y}{16^n}\right)} - 6\sqrt{f\left(\frac{x}{16^n}\right)f\left(\frac{y}{16^n}\right)} \right. \\ & \quad \left. - 4f\left(\frac{y}{16^n}\right)^{\frac{3}{4}}\sqrt[4]{f\left(\frac{x}{16^n}\right)} - f\left(\frac{y}{16^n}\right) \right\| \\ &\leq \lim_{n \rightarrow +\infty} 16^n \varphi\left(\frac{x}{16^n}, \frac{y}{16^n}\right) \\ &\leq \lim_{n \rightarrow +\infty} 16^n \times \frac{L^n}{16^n} \varphi(x, y) \\ &= 0 \end{aligned}$$

for all $x, y \in A^+$. So

$$A\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) = A(x) + 4A(x)^{\frac{3}{4}}\sqrt[4]{A(y)}$$

$$+6\sqrt{A(x)A(y)} + 4A(y)^{\frac{3}{4}}\sqrt[4]{A(x)} + A(y)$$

for all $x, y \in A^+$. Thus the mapping $A : A^+ \rightarrow B^+$ is positive-additive, as desired. $\square$

Corollary 2.3. *Let $p > 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \rightarrow B^+$ be a mapping such that*

$$\begin{aligned} & \left\| f\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) - f(x) \right. \\ & \quad \left. - 4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)} - 6\sqrt{f(x)f(y)} - 4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)} - f(y) \right\| \\ & \leq \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \end{aligned} \quad (7)$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \rightarrow B^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{(2\theta_1 + \theta_2)\|x\|^p}{16^p - 16}$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 16^{1-p}$ and we get the desired result. $\square$

Theorem 2.4. *Let $\varphi : A^+ \times A^+ \rightarrow [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{16}, \frac{y}{16}\right)$$

for all $x, y \in A^+$. Let $f : A^+ \rightarrow B^+$ be a mapping satisfying (3). Then there exists a unique positive-additive mapping $A : A^+ \rightarrow A^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{\varphi(x, x)}{16 - 16L}$$

for all $x \in A^+$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : X \rightarrow X$ such that

$$Jg(x) := \frac{1}{16}g(16x)$$

for all $x \in A^+$.

It follows from (5) that

$$\left\| f(x) - \frac{f(16x)}{16} \right\| \leq \frac{1}{16}\varphi(x, x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{1}{16}$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $0 < p < 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \rightarrow B^+$ be a mapping satisfying (7). Then there exists a unique positive-additive mapping $A : A^+ \rightarrow B^+$ satisfying (1) and*

$$\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{16 - 16^p} \|x\|^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 16^{p-1}$ and we get the desired result. $\square$

3. Stability of Eq. (1): Direct Method

In this section, using the direct method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in C^* -algebras.

Theorem 3.1. *Let $f : A^+ \rightarrow B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \rightarrow [0, \infty)$ satisfying (3) and*

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 16^j \varphi\left(\frac{x}{16^j}, \frac{y}{16^j}\right) < \infty \quad (8)$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \rightarrow A^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{1}{16} \tilde{\varphi}(x, x) \quad (9)$$

for all $x \in A^+$.

Proof. It follows from (5) that

$$\left\| f(x) - 16f\left(\frac{x}{16}\right) \right\| \leq \varphi\left(\frac{x}{16}, \frac{x}{16}\right)$$

for all $x \in A^+$. Hence

$$\left\| 16^l f\left(\frac{x}{16^l}\right) - 16^k f\left(\frac{x}{16^k}\right) \right\| \leq \frac{1}{16} \sum_{j=l+1}^k 16^j \varphi\left(\frac{x}{16^j}, \frac{x}{16^j}\right) \quad (10)$$

for all nonnegative integers k and l with $k > l$ and all $x \in A^+$. It follows from (8) and (10) that the sequence $\{16^j f(\frac{x}{16^j})\}$ is Cauchy for all $x \in A^+$. Since B^+ is complete, the sequence $\{16^j f(\frac{x}{16^j})\}$ converges. So one can define the mapping $A : A^+ \rightarrow B^+$ by

$$A(x) := \lim_{j \rightarrow \infty} 16^j f\left(\frac{x}{16^j}\right)$$

for all $x \in A^+$. By (3) and (8),

$$\begin{aligned} & \left\| A\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) - A(x) \right. \\ & \quad \left. - 4A(x)^{\frac{3}{4}}\sqrt[4]{A(y)} - 6\sqrt{A(x)A(y)} - 4A(y)^{\frac{3}{4}}\sqrt[4]{A(x)} - A(y) \right\| \\ &= \lim_{n \rightarrow +\infty} \left\| 16^n \left[f\left(\frac{x}{16^n} + 4\sqrt[4]{\frac{x^3y}{65536^n}} + 6\sqrt{\frac{xy}{256^n}} + 4\sqrt[4]{\frac{xy^3}{65536^n}} + \frac{y}{16^n}\right) \right. \right. \\ & \quad \left. - f\left(\frac{x}{16^n}\right) - 4f\left(\frac{x}{16^n}\right)^{\frac{3}{4}}\sqrt[4]{f\left(\frac{y}{16^n}\right)} - 6\sqrt{f\left(\frac{x}{16^n}\right)f\left(\frac{y}{16^n}\right)} \right. \\ & \quad \left. - 4f\left(\frac{y}{16^n}\right)^{\frac{3}{4}}\sqrt[4]{f\left(\frac{x}{16^n}\right)} - f\left(\frac{y}{16^n}\right) \right\| \\ &\leq \lim_{n \rightarrow +\infty} 16^n \varphi\left(\frac{x}{16^n}, \frac{y}{16^n}\right) \\ &= 0 \end{aligned}$$

for all $x, y \in A^+$. So

$$\left\| A \left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y \right) - A(x) - 4A(x)^{\frac{3}{4}} \sqrt[4]{A(y)} - 6\sqrt{A(x)A(y)} - 4A(y)^{\frac{3}{4}} \sqrt[4]{A(x)} - A(y) \right\| = 0$$

for all $x, y \in A^+$. Hence the mapping $A : A^+ \rightarrow B^+$ is positive-additive. Moreover, letting $l = 0$ and passing the limit $k \rightarrow \infty$ in (10), we get (9). So there exists a positive-additive mapping $A : A^+ \rightarrow B^+$ satisfying (1) and (9).

Now, let $B : A^+ \rightarrow B^+$ be another positive-additive mapping satisfying (1) and (9). Then we have

$$\begin{aligned} \|A(x) - B(x)\| &= 16^q \left\| A \left(\frac{x}{16^q} \right) - B \left(\frac{x}{16^q} \right) \right\| \\ &\leq 16^q \left\| A \left(\frac{x}{16^q} \right) - f \left(\frac{x}{16^q} \right) \right\| + 16^q \left\| B \left(\frac{x}{16^q} \right) - f \left(\frac{x}{16^q} \right) \right\| \\ &\leq 2 \cdot 16^{q-1} \tilde{\varphi} \left(\frac{x}{16^q}, \frac{x}{16^q} \right), \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in A^+$. So we can conclude that $A(x) = B(x)$ for all $x \in A^+$. This proves the uniqueness of $\mathbf{A}$. $\square$

Corollary 3.2. *Let $p > 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \rightarrow B^+$ be a mapping satisfying (7). Then there exists a unique positive-additive mapping $A : A^+ \rightarrow B^+$ satisfying (1) and*

$$\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{16^p - 16} \|x\|^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.1. Then we get the desired result. $\square$

Theorem 3.3. *Let $f : A^+ \rightarrow B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \rightarrow [0, \infty)$ satisfying (3) such that*

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{\varphi(16^j x, 16^j y)}{16^j} < \infty$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \rightarrow B^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{1}{16} \tilde{\varphi}(x, x)$$

for all $x \in A^+$.

Proof. It follows from (5) that

$$\left\| f(x) - \frac{f(16x)}{16} \right\| \leq \frac{1}{16} \varphi(x, x)$$

for all $x \in A^+$. The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Corollary 3.4. Let $0 < p < 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \rightarrow B^+$ be a mapping satisfying (7). Then there exists a unique positive-additive mapping $A : A^+ \rightarrow B^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{16 - 16^p} \|x\|^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.3. Then we get the desired result. $\square$

References

- [1] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc., Japan*, 2 (1950), 64-66.
- [2] H. Azadi Kenary, On the stability of a cubic functional equation in random normed spaces, *J. Math. Extension*, 4(1) (2009), 105-113.
- [3] H. Azadi Kenary, On the Hyers-Ulam-Rassias stability of a functional equation in non-Archimedean and random normed spaces, *Acta Univ. Apul.*, 27 (2011), 173-186.
- [4] H. Azadi Kenary, Stability of a Pexiderial Functional Equation in Random Normed Spaces, *Rend. Circ. Mat., Palermo*, (2011), 59-68.

- [5] H. Azadi Kenary and Y. J. Cho, Stability of mixed additive-quadratic Jensen type functional equation in various spaces, *Computer and Mathematics with Applications*, doi:10.1016/j.camwa.2011.03.024.
- [6] H. Azadi Kenary, Sun Young Jang and Choonkil Park, A fixed point approach to the Hyers-Ulam stability of a functional equation in various normed spaces, *Fixed Point Theory and Applications*, 2011, 2011:67, doi:10.1186/1687-1812-2011-67.
- [7] H. Azadi Kenary, Jung Rye Lee and Choonkil Park, Nonlinear approximation of an ACQ-functional equation in nan-spaces, *Fixed Point Theory and Applications*, 60 (2011), 1-22.
- [8] H. Azadi Kenary, Choonkil Park and Jung Rye Lee, Non-Archimedean stability of an AQQ functional equation, *J. Comp. Anal. Appl.*, (in press).
- [9] H. Azadi Kenary Kh. Shafaat, M. Shafei and G. Takbiri, Hyers-Ulam-Rassias Stability of the Apollonius type Quadratic Mapping in RN-Spaces, *J. Nonlinear Sci. Appl.*, 4 (2011), 110-119.
- [10] J. A. Baker, The stability of certain functional equations, *Proc. Amer. Math. Soc.*, 112 (1991), 729-732.
- [11] L. Cădariu and V. Radu, Fixed points and the stability of Jensen's functional equation, *J. Inequal. Pure Appl. Math.*, 4 (1), Art. ID 4 (2003).
- [12] L. Cădariu and V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, *Grazer Math. Ber.*, 346 (2004), 43-52.
- [13] L. Cădariu and V. Radu, Fixed point methods for the generalized stability of functional equations in a single variable, *Fixed Point Theory and Applications*, 2008, Art. ID 749392 (2008).
- [14] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.
- [15] J. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.*, 74 (1968), 305-309.
- [16] J. Dixmier, *C*-Algebras*, North-Holland Publ. Com., Amsterdam, New York and Oxford, 1977.

- [17] G. L. Forti, Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations, *J. Math. Anal. Appl.*, 295 (2004), 127-133.
- [18] G. L. Forti, Elementary remarks on Ulam-Hyers stability of linear functional equations, *J. Math. Anal. Appl.*, 328 (2007), 109-118.
- [19] L. Găvruta, Matkowski contractions and Hyers-Ulam stability, *Bul. St. Univ. Politehnica Timisoara Mat. Fiz.*, 53 (2008), 32-35.
- [20] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, 184 (1994), 431-436.
- [21] P. Găvruta and L. Găvruta, A new method for the generalized Hyers-Ulam-Rassias stability, *Internat. J. Nonlinear Anal. Appl.*, 1 (2010), 11-18.
- [22] K. R. Goodearl, Notes on Real and Complex C^* -Algebras, *Shiva Math. Series IV*, Shiva Publ. Limited, Cheshire, England, 1982.
- [23] M. E. Gordji, C. Park, and M. B. Savadkouhi, The stability of a quartic type functional equation with the fixed point alternative, *Fixed Point Theory*, 11 (2010), 265-272.
- [24] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.*, 27 (1941), 222-224.
- [25] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
- [26] G. Isac and Th. M. Rassias, Stability of ψ -additive mappings: Applications to nonlinear analysis, *Internat. J. Math. Math. Sci.*, 19 (1996), 219-228.
- [27] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press Inc., Palm Harbor, Florida, 2001.
- [28] H. A. Kenary, *Hyers-Ulam stability of some functional equations in non-Archimedean and random normed spaces*, (preprint).
- [29] D. Mihet, The Hyers-Ulam stability for two functional equations in a single variable, *Banach J. Math. Anal. Appl.*, 2 (2008), 48-52.
- [30] D. Mihet and V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J. Math. Anal. Appl.*, 343 (2008), 567-572.

- [31] A. K. Mirmostafae, Non-Archimedean stability of quadratic equations, *Fixed Point Theory*, 11 (2010), 67-75.
- [32] M. Mirzavaziri and M. S. Moslehian, A fixed point approach to stability of a quadratic equation, *Bull. Braz. Math. Soc.*, 37 (2006), 361-376.
- [33] C. Park, Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras, *Fixed Point Theory and Applications*, 2007.
- [34] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory and Applications, 2008.
- [35] V. Radu, The fixed point alternative and the stability of functional equations, *Fixed Point Theory*, 4 (2003), 91-96.
- [36] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *Bull. Sci. Math.*, 108 (1984), 445-446.
- [37] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.*, 46 (1982), 126-130.
- [38] J. M. Rassias, Solution of the Ulam stability problem for quartic mappings, *Glas. Mat. Ser.*, 54 (1999), 243-252.
- [39] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, 72 (1978), 297-300.
- [40] I. A. Rus, *Principles and Applications of Fixed Point Theory*, Ed. Dacia, Cluj-Napoca, 1979.
- [41] I. A. Rus, Remarks on Ulam stability of the operatorial equations, *Fixed Point Theory*, 10 (2009), 305-320.
- [42] I. A. Rus, A. Petruşel, and G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008.
- [43] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.

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