

On the Epsilon Hypercyclicity of a Pair of Operators

B. Yousefi*

Payame Noor University

K. Jahedi

Islamic Azad University-Shiraz Branch

Abstract. In this paper we prove that if a pair of operators is ϵ -hypercyclic for all $\epsilon > 0$, then it is topologically transitive.

AMS Subject Classification: 47B37; 47B33.

Keywords and Phrases: hypercyclic vector, ϵ -hypercyclicity, transitive operator.

1. Introduction

From now on, let T_1, T_2 be commutative bounded linear operators on an infinite dimensional Banach space X and consider the pair $T = (T_1, T_2)$.

Definition 1.1. Put $\mathcal{F} = \{T_1^m T_2^n : m, n \geq 0\}$. For $x \in X$, the orbit of x under T is the set $\text{Orb}(T, x) = \{Sx : S \in \mathcal{F}\}$. The vector x is called a hypercyclic vector for the pair T if $\text{Orb}(T, x)$ is dense in X .

Definition 1.2. We say that the pair $T = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets U and V of X there exists $S \in \mathcal{F}$ such that $S(U) \cap V \neq \emptyset$.

Definition 1.3. Let $\epsilon \in (0, 1)$ and $x \in X$. If for every non-zero vector $y \in X$, there exist integers m, n such that

$$\|T_1^m T_2^n x - y\| < \epsilon \|y\|,$$

Received December 2010; Final Revised September 2011

*Corresponding author

then the vector x is called ϵ -hypercyclic for the pair $T = (T_1, T_2)$. A pair of operators is ϵ -hypercyclic if it admits an ϵ -hypercyclic vector.

For some sources on these topics see [1–18].

2. Main Results

In this section we prove that if a pair is ϵ -hypercyclic for all $\epsilon > 0$, then it is topologically transitive and consequently it is hypercyclic. We will extend Theorem 1.4 in [1] for a pair of operators and we will use the idea of it's proof. We will denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

Theorem 2.1. *Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators T_1 and T_2 . If for every $\epsilon > 0$, T is ϵ -hypercyclic, then T is topologically transitive.*

Proof. Suppose that U and V are nonempty open subsets of X . Let $u \in U$ and $v \in V$ be two nonzero vectors, and consider

$$0 < \delta < \min\{\|u\|, \|v\|\}$$

small enough such that $B(u, \delta) \subset U$ and $B(v, \delta) \subset V$. Choose

$$\epsilon < \delta / (6 \max\{\|u\|, \|v\|\}),$$

and let $x \in X$ be an ϵ -hypercyclic vector for T . This implies that there exist nonnegative integers m_0 and n_0 such that

$$\|T_1^{m_0} T_2^{n_0} x - u\| < \epsilon \|u\| < \delta.$$

Hence

$$T_1^{m_0} T_2^{n_0} x \in B(u, \delta) \subset U.$$

We want to show that

$$V \cap \{T_1^m T_2^n x : m, n \in \mathbb{N}_0\}$$

contain infinitely many elements. Suppose on the contrary that it contains only the elements $T_1^{n_i^{(1)}} T_2^{n_i^{(2)}} x$ for $i = 1, \dots, k$. As we saw earlier, for each $v' \in B(v, \frac{2\delta}{3})$ there exist integers $m(v')$ and $n(v')$ which satisfies

$$\| T_1^{m(v')} T_2^{n(v')} x - v' \| \leq \epsilon \| v' \| \leq 2\epsilon \| v \| < \frac{\delta}{3}.$$

Hence

$$T_1^{m(v')} T_2^{n(v')} x \in \{T_1^{n_i^{(1)}} T_2^{n_i^{(2)}} x : i = 1, \dots, k\},$$

because

$$\| T_1^{m(v')} T_2^{n(v')} x - v \| \leq \| T_1^{m(v')} T_2^{n(v')} x - v' \| + \| v' - v \| < \delta.$$

Therefore

$$B(v, \frac{2\delta}{3}) \subset \bigcup_{i=1}^k B(T_1^{n_i^{(1)}} T_2^{n_i^{(2)}} x, \frac{\delta}{3}),$$

that is a contradiction since X is infinite dimensional. Thus indeed the set

$$V \cap \{T_1^m T_2^n x : m, n \in \mathbb{N}_0\}$$

contain infinitely many elements and so the set

$$B(v, \delta) \cap \{T_1^m T_2^n x : m, n \in \mathbb{N}_0\}$$

has infinite elements. In particular, there exist $m, n \in \mathbb{N}_0$ satisfying $m \geq m_0$ and $n \geq n_0$ such that $T_1^m T_2^n x \in V$. Thus we get

$$T_1^{m-m_0} T_2^{n-n_0} T_1^{m_0} T_2^{n_0} x = T_1^m T_2^n x,$$

which belongs to

$$T_1^{m-m_0} T_2^{n-n_0}(U) \cap V.$$

This completes the proof. $\square$

In the proof of the following lemma, we use a method of the proof of Theorem 1.2 in [5] to extend the results for tuples. We will use $\text{HC}(T)$ for the collection of hypercyclic vectors for the pair of operator T .

Lemma 2.2. *Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators T_1 and T_2 . Then T is topologically transitive if and only if $HC(T)$ is dense in X .*

Proof. Fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of the open balls in X with rational radii, and centers in a countable dense subset of X . By the continuity of the operators T_1 and T_2 , the sets

$$G_k = \bigcup \{T_1^{-m}T_2^{-n}(B_k) : m, n \in \mathbb{N}_0\}$$

are open. Clearly $HC(T)$ is equal to

$$\bigcap \{G_k : k \in \mathbb{N}\}.$$

Now let T be topologically transitive and let W be an arbitrary nonempty open set in X . Then for all $k \in \mathbb{N}$, there exist $m(k)$ and $n(k)$ in $\mathbb{N}$ such that

$$T_1^{m(k)}T_2^{n(k)}W \cap B_k \neq \emptyset$$

which implies that $W \cap G_k \neq \emptyset$ for all k . Thus each G_k is dense in X and so by the Baire Category Theorem $HC(T)$ is also dense in X . Conversely, if $HC(T)$ is dense in X , then each set G_k is so. This implies clearly that T is topologically transitive.

Corollary 2.3. *Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators T_1 and T_2 . If for every $\epsilon > 0$, T is ϵ -hypercyclic, then T is hypercyclic.*

Proof. If for every $\epsilon > 0$, T is ϵ -hypercyclic then by Theorem 2.1, T is topologically transitive. Now, by the Lemma 2.2, $HC(T)$ is dense in X and this implies clearly that the pair T is hypercyclic. $\square$

References

- [1] C. Badea, S. Grivaux, and V. Muller, *Epsilon-hypercyclic operators*, Institute of Mathematics, AS CR, Prague, (2008), 7-24.
- [2] J. Bes and A. Peris, *Hereditarily hypercyclic operators*, J. Func. Anal., no.1, 167 (1999), 94-112.
- [3] P. S. Bourdon and J. H. Shapiro, Hypercyclic operators that commute with the Bergman backward shift, *Trans. Amer. Math. Soc.*, no.11, 352 (2000), 5293-5316.
- [4] R. M. Gethner and J. H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.*, 100 (1987), 281-288.
- [5] G. Godefroy and J. H. Shapiro, Operators with dense, invariant cyclic vector manifolds, *Journal of Functional Analysis*, 98 (1991), 229-269.
- [6] K. Goswin and G. Erdmann, Universal families and hypercyclic operators, *Bulletin of the American Mathematical Society*, 35 (1999), 345-381.
- [7] H. N. Salas, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.*, 347 (1995), 993-1004.
- [8] S. Shkarin, Non-sequential weak supercyclicity and hypercyclicity, *Journal of Functional Analysis*, 242 (1) (2007), 37-77.
- [9] B. Yousefi and H. Rezaei, Hypercyclicity on the algebra of Hilbert-Schmidt operators, *Results in Mathematics*, 46 (2004), 174-180.
- [10] B. Yousefi and H. Rezaei, Some necessary and sufficient conditions for Hypercyclicity Criterion, *Proc. Indian Acad. Sci. (Math. Sci.)*, 115, No. 2 (2005), 209-216.
- [11] B. Yousefi and A. Farrokhinia, On the hereditarily hypercyclic vectors, *Journal of the Korean Mathematical Society*, 43 (6) (2006), 1219-1229.
- [12] B. Yousefi, H. Rezaei, and J. Doroodgar, Supercyclicity in the operator algebra using Hilbert-Schmidt operators, *Rendiconti Del Circolo Matematico di Palermo*, Serie II, Tomo LVI (2007), 33-42.
- [13] B. Yousefi and H. Rezaei, Hypercyclic property of weighted composition operators, *Proc. Amer. Math. Soc.*, 135 (10) (2007), 3263-3271.

- [14] B. Yousefi and S. Haghighi, Hypercyclicity of special operators on Hilbert function spaces, *Czechoslovak Mathematical Journal*, 57 (132) (2007), 1035-1041.
- [15] B. Yousefi and H. Rezaei, On the supercyclicity and hypercyclicity of the operator algebra, *Acta Mathematica Sinica*, 24 (7) (2008), 1221-1232.
- [16] B. Yousefi and R. Soltani, Hypercyclicity of the adjoint of weighted composition operators, *Proc. Indian Acad. Sci. (Math. Sci.)*, 119 (3) (2009), 513-519.
- [17] B. Yousefi and J. Izadi, Weighted composition operators and supercyclicity criterion, *International Journal of Mathematics and Mathematical Sciences*, Volume 2011, DOI 10.1155/2011/514370 (2011).
- [18] B. Yousefi, Hereditarily transitive tuples, *Rend. Circ. Mat. Palermo*, DOI 10.1007/S12215-011-0066-y (2011).

Bahmann Yousefi

Department of Mathematics
Professor of Mathematics
Payame Noor University
P.O. Box 19395-3697
Tehran, Iran
E-mail: b_yousefi@pnu.ac.ir

Khadije Jahedi

Department of Mathematics
Assistant Professor of Mathematics
Islamic Azad University-Shiraz Branch
Shiraz, Iran
E-mail: mjahedi80@yahoo.com