Generalized Composition Operators Between
Weighted Dirichlet Type Spaces and
Bloch Type Spaces

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Abstract. We characterize bounded and compact generalized composition operators between Bloch type spaces and weighted Dirichlet type spaces. Then, we show that these results can be employed to characterize bounded and compact Volterra type operators between these spaces. Finally, we give a connection between weighted and generalized composition operators to conclude some results about boundedness and compactness of them.

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1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. For $\alpha \in (0, \infty)$, the Bloch type space (or $\alpha$-Bloch space) $B^\alpha$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$
\|f\|_{B^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.
$$

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The little Bloch type space $\mathcal{B}_0^\alpha$ consists of those functions $f \in \mathcal{B}^\alpha$ for which
\[
\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.
\]
The space $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}$ and $\mathcal{B}_0^\alpha$ is a closed subspace of $\mathcal{B}^\alpha$. If $\alpha = 1$ or $\alpha \in (0, 1)$, then $\mathcal{B}^\alpha$ coincides with the classical Bloch space $\mathcal{B}$ or the classical $(1 - \alpha)$-Lipschitz class $\Lambda_{1-\alpha}$, respectively (see [5, Theorem 5.1] or [20]).

For $p \in (0, \infty)$ and $\beta > -1$, the weighted Bergman space $A_p^\beta$ is the space of all $f \in \mathcal{H}({\mathbb{D}})$ for which
\[
\|f\|_{A_p^\beta}^p = \int_{\mathbb{D}} |f(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \infty,
\]
where $dA$ denotes normalized Lebesgue area measure on $\mathbb{D}$. Equivalently, $f \in A_p^\beta$ if and only if
\[
\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\beta dA(z) < \infty.
\]

It is well known that $A_p^\beta$ is a Banach space for $p \geq 1$, and a Hilbert space for $p = 2$.

For $p \in (0, \infty)$ and $\beta > -1$, the weighted Dirichlet type space $D_p^\beta$ is the space of all $f \in \mathcal{H}({\mathbb{D}})$ satisfying
\[
\|f\|_{D_p^\beta}^p = \int_{\mathbb{D}} |f'(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \infty.
\]

We note that $f \in D_p^\beta$ if and only if $f' \in A_p^\beta$. It is well known that $f \in A_p^\beta$ if and only if $f' \in A_{p+\beta}^\beta$. Namely
\[
\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\beta dA(z) \simeq \int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p+\beta} dA(z) + |f(0)|^p \quad (1)
\]

for $p \in (0, \infty), \beta > -1$ and for any analytic function $f$ in $\mathbb{D}$ which is a standard result. The inequality in one direction is a classical result due to Hardy and Littlewood (see [5, Theorem 5.6]), while the reverse inequality can easily be proved by the methods used in [5, Chapter 5].
Therefore one can say, $D^{\beta}_{p+\beta} = A^{\beta}_{p}$. For general background about weighted Bergman spaces $A^{\beta}_{p}$ and Bloch type spaces $B^\alpha$ we refer [7], [20] and the references therein.

In (1) and in the sequel of this paper, we use the notation $A \lesssim B$ to mean that there is a positive constant $C$ such that $A \leq CB$, and $A \simeq B$ if $A \lesssim B \lesssim A$. Constants denoted by $C$, are positive and not necessarily the same in each occurrence.

Let $\varphi$ be a non constant analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. The weighted composition operator $gC_\varphi$ is defined by $gC_\varphi(f) = g(f \circ \varphi)$ for $f \in H(\mathbb{D})$. In the special case of $g = 1$, we get the composition operator $C_\varphi(f) = f \circ \varphi$. The generalized composition operator defined by

$$(C^g_\varphi f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(\mathbb{D}),$$

was introduced by Li and Stević in [8]. If $g = \varphi'$ then $C^g_\varphi$ is essentially composition operator, since the difference $C^g_\varphi - C_\varphi$ is a constant.

Therefore, $gC_\varphi$ and $C^g_\varphi$ are generalizations of the composition operators.

The main subject in the study of (weighted) (generalized) composition operators is to describe operator theoretic properties of $(gC_\varphi)(C^g_\varphi) C_\varphi$ in terms of function theoretic properties of $(g) \varphi$. For general theory of composition operators, we refer the reader to [4] and [16]. Boundedness and compactness of composition operators $C_\varphi$ on Bloch space $\mathcal{B}$ were described by Madigan and Matheson [10]. Recently, Ohno, Stroethoff and Zhao studied weighted composition operators between Bloch type spaces in [13]. Also composition operators from Bloch space $\mathcal{B}$ to the weighted Dirichlet space $D^2_\beta$ were considered by Smith in [12]. Li and Stević investigated the boundedness and compactness of the generalized composition operators on Bloch type spaces and Zygmund spaces in [8]. The authors investigated boundedness and compactness of the generalized composition operators and the products of Volterra type operators and composition operators between $Q_K$ spaces in [11] and between logarithmic Bloch type spaces and $Q_K'$ type spaces in [15].

In this paper we are interested in the problem of using function theoretic properties of analytic functions $g$ and analytic self-maps $\varphi$ on $\mathbb{D}$ to determine when generalized composition operator $C^g_\varphi$ between the weighted...
Dirichlet type spaces and Bloch type spaces is bounded or compact. As the following equations show, these results can be employed to characterize bounded and compact Volterra type operators between the above mentioned spaces where the Volterra type operators defined on \( \mathcal{H}(\mathbb{D}) \) as follows

\[
V_g(f)(z) = \int_0^z f(\xi)g'(\xi)\,d\xi \quad \text{and} \quad U_g(f)(z) = \int_0^z f'(\xi)g(\xi)\,d\xi,
\]

for symbol \( g \) in \( \mathcal{H}(\mathbb{D}) \). If \( g(z) = z \) or \( g(z) = \log \frac{1}{1-z} \), then \( V_g \) is the integration operator or the Cesàro operator, respectively. For a survey on the study of such operators see [17]. Note that

\[
U_gC_\varphi(f) = \int_0^z (f \circ \varphi)'(\xi)g(\xi)\,d\xi = C_{\varphi'g}(f),
\]

so \( U_g \) is the generalized composition operator \( C^g_\varphi \) for \( \varphi(z) = z \). Therefore, \( U_g \) is bounded (compact) whenever \( C^g_\varphi \) is bounded (compact) when \( \varphi(z) = z \). As we know, on a general space of analytic functions, the differentiation operator \( D \) is typically unbounded. On the other hand, the composition operator \( C_\varphi \) is bounded on various spaces of analytic functions on \( \mathbb{D} \) (see [5, 16]), though the product \( DC_\varphi \) is possibly still unbounded there. Here, we consider new operator \( V_gDC_\varphi \), noting the equality

\[
V_gDC_\varphi(f) = \int_0^z (f \circ \varphi)'(\xi)g'(\xi)\,d\xi = C^{\varphi'g}_\varphi(f),
\]

one can consider \( V_gDC_\varphi \) as a generalized composition operator, and apply the results obtained about generalized composition operators to characterize boundedness and compactness of \( V_gDC_\varphi \). Finally, we give a connection between weighted composition operators \( gC_\varphi \) and generalized composition operators \( C^g_\varphi \) and conclude some results about boundedness and compactness of them.

In the sequel, we need the following estimates. For functions \( f \) in \( B^\alpha \),
integrating the estimate \( |f'(z)| \leq \frac{\|f\|_{B^\alpha}}{(1-|z|^2)^{\alpha}} \), we obtain

\[
|f(z) - f(0)| = \left| \int_0^z f'(\zeta) d\zeta \right| \leq \|f\|_{B^\alpha} \int_0^{|z|} \frac{d\rho}{(1-\rho^2)^{\alpha}} \\
\leq \frac{1}{(1-|z|^2)^{\alpha}} \|f\|_{B^\alpha}. \tag{2}
\]

To estimate functions in \( D_p^\beta \), we make use of the following lemma. For the proof see [6], [7], or the original source [19].

**Lemma 1.1.** Let \( 0 < p < \infty, \beta > -1 \) and \( f \in A_p^\beta \). Then

\[
|f(z)|(1-|z|^2)^{\frac{2+\beta}{p}} \leq \left(1 + \beta\right) \int_{D} |f(z)|^p (1-|z|^2)^{\beta} dA(z) \right)^{\frac{1}{p}} \quad (z \in D),
\]

with equality if and only if \( f \) is a constant multiple of the function \( f_a(z) = (-\sigma'_a(z))^{\frac{2+\beta}{p}} \).

From this Lemma, we have

\[
|f'(z)| \leq C \frac{\|f\|_{D_p^\beta}}{(1-|z|^2)^{\frac{2+\beta}{p}}} \quad (z \in D).
\]

for \( f \in D_p^\beta \) and hence

\[
|f(z) - f(0)| = \left| \int_0^z f'(\zeta) d\zeta \right| \leq C \|f\|_{D_p^\beta} \int_0^{|z|} \frac{d\rho}{(1-\rho^2)^{\frac{2+\beta}{p}}} \\
\leq \frac{C}{(1-|z|^2)^{\frac{2+\beta}{p}}} \|f\|_{D_p^\beta}. \tag{3}
\]

Using the estimates (2) and (3), it follows from the proof of the Weak Convergence Theorem in [16] that:

**Lemma 1.2.** Let \( 0 < \alpha, p < \infty, \beta > -1, g \in \mathcal{H}(D) \) and \( \varphi \) be an analytic self-map of \( D \). Let \( X = B_\alpha^\beta \) or \( D_p^\beta; Y = B_\alpha^\beta \) or \( D_p^\beta \). Then \( C_g^\beta : X \to Y \) is compact if and only if for any bounded sequence \( (f_n) \) in \( X \) which converges to zero uniformly on compact subsets of \( D \) as \( n \to \infty \), we have \( \|C_g^\beta f_n\|_Y \to 0 \) as \( n \to \infty \).
2. The Boundedness and Compactness of $C_\varphi^g : \mathcal{B}^\alpha \rightarrow \mathcal{D}^p_\beta$

In this section, we characterize the boundedness and compactness of the generalized composition operators from Bloch type spaces $\mathcal{B}^\alpha$ to weighted Dirichlet type spaces $\mathcal{D}^p_\beta$. To do this, we need $\alpha$-generalized hyperbolic $\mathcal{D}^p_\beta$ denoted by $\mathcal{D}^{h,g}_{\alpha,\beta,p}$, defined as $\mathcal{D}^p_\beta$ but with the ordinary derivative $|\varphi'(z)|$ replaced by the $\alpha$-generalized hyperbolic derivative $|g(z)| \log \frac{1}{|z|}$.

Thus, for $0 < \alpha, p < \infty$, $\beta > -1$ and $g \in \mathcal{H}(\mathbb{D})$, an analytic function $\varphi$ on $\mathbb{D}$ belongs to $\mathcal{D}^{h,g}_{\alpha,\beta,p}$ if and only if

$$\|\varphi\|^p_{\mathcal{D}^{h,g}_{\alpha,\beta,p}} = \int_\mathbb{D} \frac{|g(z)|^p}{(1 - |\varphi(z)|^2)^\alpha} \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \infty.$$ 

**Theorem 2.1.** Let $0 < \alpha, p < \infty$, $\beta > -1$, $g \in \mathcal{H}(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.

(i) $\varphi \in \mathcal{D}^{h,g}_{\alpha,\beta,p}$;

(ii) $C_\varphi^g : \mathcal{B}^\alpha \rightarrow \mathcal{D}^p_\beta$ is bounded;

(iii) $C_\varphi^g : \mathcal{B}_0^\alpha \rightarrow \mathcal{D}^p_\beta$ is bounded.

**proof.** (i)$\implies$(ii). For $f \in \mathcal{B}^\alpha$, we have

$$|(C_\varphi^g f)'(z)| = |f'(\varphi(z))| |g(z)| \leq \frac{|g(z)|}{(1 - |\varphi(z)|^2)^\alpha} \|f\|_{\mathcal{B}^\alpha},$$

and then

$$\|C_\varphi^g f\|^p_{\mathcal{D}^p_\beta} = \int_\mathbb{D} |f'(\varphi(z))|^p |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) \leq \|f\|^p_{\mathcal{B}^\alpha} \|\varphi\|^p_{\mathcal{D}^{h,g}_{\alpha,\beta,p}} .$$

Thus if $\varphi \in \mathcal{D}^{h,g}_{\alpha,\beta,p}$, then $C_\varphi^g : \mathcal{B}^\alpha \rightarrow \mathcal{D}^p_\beta$ is bounded.

(ii)$\implies$(iii). It is trivial, since $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.
(iii)⇒(i). For \( f \in \mathcal{B}^\alpha \) if we set \( f_s(z) = f(sz) \) for \( 0 < s < 1 \), then \( f_s \in \mathcal{B}_0^\alpha \) and \( \|f_s\|_{\mathcal{B}_0^\alpha} \leq \|f\|_{\mathcal{B}^\alpha} \). Using a well known result of Ramey and Ullrich [14], there are functions \( f_1, f_2 \in \mathcal{B}^\alpha \) such that

\[
(1 - |z|^2)^{-\alpha} \lesssim |f'_1(z)| + |f'_2(z)| \quad (z \in \mathbb{D}).
\]

Then

\[
\int_{\mathbb{D}} \frac{|sg(z)|^p}{(1 - |z|^2)^{\alpha p}} \left( \log \frac{1}{|z|} \right)^\beta dA(z) \lesssim \int_{\mathbb{D}} \left( |f'_1(s\varphi(z))|^p + |f'_2(s\varphi(z))|^p \right) \times |sg(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z)
\]

\[= \|C_{\varphi}^g f_{1s}\|_{\mathcal{D}_{D_0}^\alpha}^p + \|C_{\varphi}^g f_{2s}\|_{\mathcal{D}_{D_0}^\alpha}^p,
\]

holds for all \( s \in (0,1) \). Hence, if \( C_{\varphi}^g : \mathcal{B}_0^\alpha \rightarrow \mathcal{D}_\beta^p \) is bounded, by an application of Fatou’s lemma, this estimate implies that \( \|\varphi\|_{\mathcal{D}_{D_0}^{h,g}}^p < \infty \) and \( \varphi \in D^{h,g}_{\alpha,\beta,p} \). The proof is now complete. \( \square \)

**Theorem 2.2.** Let \( 0 < \alpha, p < \infty, \beta > -1 \), \( g \in \mathcal{H}(\mathbb{D}) \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then the following statements are equivalent.

(i) \( C_{\varphi}^g : \mathcal{B}^\alpha \rightarrow \mathcal{D}_\beta^p \) is compact;

(ii) \( C_{\varphi}^g : \mathcal{B}_0^\alpha \rightarrow \mathcal{D}_\beta^p \) is compact;

(iii) \( \lim_{t \rightarrow 1} \int_{|\varphi(z)| > t} \frac{|g(z)|^p}{(1 - |z|^2)^{\alpha p}} \left( \log \frac{1}{|z|} \right)^\beta dA(z) = 0 \) and \( \varphi \in D^{h,g}_{\alpha,\beta,p} \).

**proof.** (iii)⇒(i). From the boundedness of \( C_{\varphi}^g \) with \( f(z) = z \), we have

\[
L := \int_{\mathbb{D}} |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \infty.
\]

Let \( (f_n) \) be a sequence in the closed unit ball of \( \mathcal{B}^\alpha \) such that \( f_n \rightarrow 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( n \rightarrow \infty \). By hypothesis, for every \( \varepsilon > 0 \), there is \( \delta \in (0,1) \) such that

\[
\int_{|\varphi(z)| > \delta} \frac{|g(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p}} \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \varepsilon.
\]

Let \( K = \{ w \in \mathbb{D} : |w| \leq \delta \} \). Then
\[ \| C^\varphi f_n \|_{D_\beta}^p = \int_D |f'_n(\varphi(z))|^p |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) \]
\[ = \int_{|\varphi(z)| \leq \delta} |f'_n(\varphi(z))|^p |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) \]
\[ + \int_{\delta < |\varphi(z)| < 1} |f'_n(\varphi(z))|^p |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) \]
\[ \leq L \sup_{w \in K} |f'_n(w)|^p + \varepsilon \| f_n \|_{B^\alpha}^p \leq L \sup_{w \in K} |f'_n(w)|^p + \varepsilon. \]

By [3, IV, Theorem 2.1], the sequence \((f'_n)\) also converges to zero on compact subsets of \(D\) as \(n \to \infty\). In particular, \(\sup_{w \in K} |f'_n(w)|^p \to 0\) as \(n \to \infty\). Hence \(\| C^\varphi f_n \|_{D_\beta}^p \to 0\) as \(n \to \infty\). By Lemma 1.2, \(C^\varphi : B^\alpha \to D_\beta\) is compact.

(i) \(\implies\) (ii). It is trivial, since \(B^\alpha_0 \subset B^\alpha\).

(ii) \(\implies\) (iii). Suppose \(C^\varphi : B^\alpha_0 \to D_\beta\) is compact. Then by Theorem 2.1, \(\varphi \in D^h_\alpha\). Next, since \((\frac{z^n}{n^{1-\alpha}})\) is norm bounded in \(B^\alpha_0\) and it converges to zero uniformly on compact subsets of \(D\), using Lemma 1.2, we have \(\| C^\varphi (\frac{z^n}{n^{1-\alpha}}) \|_{D_\beta}^p \to 0\) as \(n \to \infty\). Whence for every \(\varepsilon > 0\) there is an integer \(N > 1\) such that

\[ n^{\alpha p} \int_D |\varphi(z)|^{p(n-1)} |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \varepsilon, \]

for all \(n \geq N\). Thus for each \(r \in (0,1)\)

\[ N^{\alpha p} r^{p(N-1)} \int_{|\varphi(z)| > r} |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \varepsilon. \]

Taking \(r \geq N^{\frac{\alpha}{1-r}}\), we obtain

\[ \int_{|\varphi(z)| > r} |g(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \varepsilon. \] 

(4)
On the other hand, for \( f \in \mathbb{B}_{B_0} \), the unit ball of \( B_0^\alpha \), if we let \( f_t(z) = f(tz) \), then \( f_t \to f \) uniformly on compact subsets of \( \mathbb{D} \) as \( t \to 1 \). Since \( C_\varphi : \mathbb{B}_0^\alpha \to \mathcal{D}_\beta^p \) is compact, using Lemma 1.2, \( \|C_\varphi^g(f_t - f)\|_{\mathcal{D}_\beta^p} \to 0 \) as \( t \to 1 \). Thus for every \( \varepsilon > 0 \) there is a \( t \in (0, 1) \) such that

\[
\int_{\mathbb{D}} |(C_\varphi^g f_t)'(z) - (C_\varphi^g f)'(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \varepsilon.
\]

Using this inequality along with (4), we have

\[
\int_{|\varphi(z)| > r} |(C_\varphi^g f)'(z)|^p (\log \frac{1}{|z|})^\beta dA(z) \\
\leq C \varepsilon + C \int_{|\varphi(z)| > r} |(C_\varphi^g f_t)'(z)|^p (\log \frac{1}{|z|})^\beta dA(z) \\
\leq C \varepsilon (1 + \sup_{z \in \mathbb{D}} |f_t'(z)|^p).
\]

Thus for every \( f \in \mathbb{B}_{B_0} \) and every \( \varepsilon > 0 \), there exists a \( \delta = \delta(f, \varepsilon) \) such that

\[
\int_{|\varphi(z)| > r} |(C_\varphi^g f)'(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < \varepsilon,
\]

for all \( r \in (\delta, 1) \). Since \( C_\varphi^g \) is compact, thus \( \{C_\varphi^g f : f \in \mathbb{B}_{B_0}\} \) is a relatively compact subset of \( \mathcal{D}_\beta^p \). Hence there exist finitely many functions \( f_1, \ldots, f_m \in \mathbb{B}_{B_0} \) such that for each \( f \in \mathbb{B}_{B_0} \), \( \|C_\varphi^g f - C_\varphi^g f_k\|_{\mathcal{D}_\beta^p} < \varepsilon \) for some \( k \in \{1, \ldots, m\} \). Let \( \delta = \max_{1 \leq k \leq m} \delta(f_k, \varepsilon) \). Then for each \( f \in \mathbb{B}_{B_0} \) we have

\[
\int_{|\varphi(z)| > r} |(C_\varphi^g f)'(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) < C \varepsilon,
\]

if \( r \in (\delta, 1) \). As mentioned in the previous theorem, there are two functions \( f_1, f_2 \in \mathcal{B}_\alpha \) such that

\[
(1 - |z|^2)^{-\alpha} \lesssim |f_1'(z)| + |f_2'(z)|.
\]

If \( f_{1s}, f_{2s} \) defined as in the previous theorem, then we have
\[ \varepsilon > C \int_{|\varphi(z)| > r} \left[ \frac{1}{\|f_1\|_\infty} |(C_\beta^p f_1 \circ z)(z)|^p + \frac{1}{\|f_2\|_\infty} |(C_\beta^p f_2 \circ z)'(z)|^p \right] \left( \log \frac{1}{|z|} \right)^\beta dA(z) \]
\[ \geq C \int_{|\varphi(z)| > r} \left[ |f_1'(s \varphi(z))|^p + |f_2'(s \varphi(z))|^p \right] |s \varphi(z)|^p \left( \log \frac{1}{|z|} \right)^\beta dA(z) \]
\[ \geq C \int_{|\varphi(z)| > r} \frac{|s \varphi(z)|^p}{1 - |s \varphi(z)|^2} | \log \frac{1}{|z|} |^\beta dA(z), \]

for all \( r \in [\delta, 1) \). Using Fatou’s lemma, this estimate implies (iii). \( \square \)

We recall that for \( p \geq 1 \), \( D_\beta^p \) is a Banach space. Using Lemma 4 in [8], one can say \( C_\beta^p : B_0^\alpha \rightarrow D_\beta^p \) is weakly compact if and only if \( C_\beta^p : B_0^\alpha \rightarrow D_0^p \) is compact.

In the case \( p = 2 \) and \( \alpha = 1 \), we obtain the following result. First we define \( \sigma_a(z) = \frac{a - z}{1 - \overline{a}z} \) for \( a \in \mathbb{D} \), the conformal automorphism of \( \mathbb{D} \) which exchanges the origin and the point \( a \). Such a map is its own inverse and satisfies the fundamental identities

\[ |\sigma_a'(z)| = \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \quad \text{and} \quad 1 - |\sigma_a(z)|^2 = (1 - |z|^2)|\sigma_a'(z)|. \]

**Theorem 2.3.** Let \( \beta > 0 \), \( g \in \mathcal{H}(\mathbb{D}) \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). If \( C_\beta^p : \mathcal{B} \rightarrow D_\beta^p \) is compact, then

\[ \lim_{|\varphi(z)| \rightarrow 1} \frac{|g(z)|(1 - |z|)^{\frac{\beta}{2} + 1}}{1 - |\varphi(z)|^2} = 0. \]

The converse of this implication is true, provided that

\[ L := \int_{|\varphi(z)| > r} \frac{dA(z)}{(1 - |z|^2)^2} < \infty, \text{ for } r \in (0, 1). \]

**proof.** Let \( f_a(z) = \frac{1 - |a|^2}{a} (2 + \frac{1}{1 - az}) \) for any \( a \in \mathbb{D} \setminus \{0\} \). Clearly

\[ |f_a'(z)| = \frac{1 - |a|^2}{|1 - az|^2} = |\sigma_a'(z)| = \frac{1 - |\sigma_a(z)|^2}{1 - |z|^2} \leq \frac{1}{1 - |z|^2} \]

which implies that \( \sup_{a \in \mathbb{D}} |f_a'(z)|(1 - |z|^2) \leq 1 \). Also \( f_a(0) = 3 \frac{1 - |a|^2}{a} \). So \( \{ f_a : \frac{1}{2} \leq |a| < 1 \} \) is a bounded family of functions in \( \mathcal{B} \) and \( f_a \) converges
to zero uniformly on compact subsets of \( \mathbb{D} \) as \( |a| \to 1 \). By compactness of \( C^g_\varphi \) and Lemma 1.2, we have \( \|C^g_\varphi f_a\|_{D^2_\beta} \to 0 \) as \( |a| \to 1 \). Let \( D(\lambda, r) \) denote the pseudohyperbolic disk \( D(\lambda, r) = \{ z \in \mathbb{D} : |\sigma_\lambda(z)| < r \} \). Observe that for any \( \lambda \in \mathbb{D} \) and \( 0 < r < 1 \), we have

\[
\|C^g_\varphi f_a\|_{D^2_\beta}^2 = \int_{\mathbb{D}} |f'_a(\varphi(z))|^2 |g(z)|^2 \left( \log \frac{1}{|z|} \right)^\beta dA(z)
\]

\[
\geq \int_{D(\lambda, r)} |f'_a(\varphi(z))|^2 |g(z)|^2 \left( \log \frac{1}{|z|} \right)^\beta dA(z)
\]

\[
= \int_{|w| < r} |f'_a(\varphi(\sigma_\lambda(w)))|^2 |g(\sigma_\lambda(w))|^2 |\sigma_\lambda'(w)|^2 \left( \log \frac{1}{|\sigma_\lambda(w)|} \right)^\beta dA(w)
\]

\[
\geq C \int_{|w| < r} \left| ((C^g_\varphi f_a) \circ \sigma_\lambda)'(w) \right|^2 (1 - |\sigma_\lambda(w)|^2)^\beta dA(w)
\]

\[
\geq C \left( \frac{1 - |\lambda|}{1 + |\lambda|} \right)^\beta \int_{|w| < r} \left| ((C^g_\varphi f_a) \circ \sigma_\lambda)'(w) \right|^2 dA(w)
\]

\[
= C \left( \frac{1 - |\lambda|}{1 + |\lambda|} \right)^\beta \frac{(1 - |a|^2)^2}{|1 - \overline{a}\varphi(\lambda)|^4} |g(\lambda)|^2 (1 - |\lambda|^2)^2
\]

\[
= C \frac{(1 - |\lambda|)^{\beta + 2}}{(1 + |\lambda|)^{\beta - 2}} \frac{(1 - |a|^2)^2}{|1 - \overline{a}\varphi(\lambda)|^4} |g(\lambda)|^2.
\]

\[
(5)
\]

Since \( \|C^g_\varphi f_a\|_{D^2_\beta} \to 0 \) as \( |a| \to 1 \), then for given \( \varepsilon > 0 \) there is an \( r \), \( 0 < r < 1 \), such that \( \|C^g_\varphi f_a\|_{D^2_\beta} < \varepsilon \) whenever \( |a| > r \). Hence for any point \( z_0 \in \mathbb{D} \) with \( |\varphi(z_0)| > r \) we have \( \|C^g_\varphi f_{\varphi(z_0)}\|_{D^2_\beta} < \varepsilon \). Hence, employing (5) with \( \varphi(z_0) \) instead of \( a \), we obtain

\[
\frac{(1 - |\lambda|)^{\frac{\beta + 1}{2}}}{(1 + |\lambda|)^{\frac{\beta - 1}{2}}} \frac{1 - |\varphi(z_0)|^2}{|1 - \overline{\varphi(z_0)}\varphi(\lambda)|^2} |g(\lambda)| < \varepsilon.
\]

Setting \( \lambda = z_0 \), we have

\[
C \frac{|g(z_0)|(1 - |z_0|)^{\frac{\beta + 1}{2}}}{1 - |\varphi(z_0)|^2} < \varepsilon.
\]
Hence
\[ \lim_{|\varphi(z)| \to 1} \frac{|g(z)|(1 - |z|)^{\frac{\beta}{2} + 1}}{1 - |\varphi(z)|^2} = 0. \]

To prove the converse, assume \( L = \int_{|\varphi(z)| > r} \frac{dA(z)}{(1 - |z|)^{\beta}} < \infty \) and
\[ \lim_{|\varphi(z)| \to 1} \frac{|g(z)|(1 - |z|)^{\frac{\beta}{2} + 1}}{1 - |\varphi(z)|^2} = 0. \]

For compactness of \( C^\phi_\beta : B \to D^\beta_\alpha \), we show that the conditions in (iii) of Theorem 2.2 hold. By the definition of limit, for every \( \varepsilon > 0 \) there exists \( r \in (0, 1) \) such that
\[ \frac{|g(z)|(1 - |z|)^{\frac{\beta}{2}}}{1 - |\varphi(z)|^2} < \frac{\varepsilon}{1 - |z|}, \]
for every \( z \) with \( |\varphi(z)| > r \). Integrating this inequality, we obtain
\[ \int_{|\varphi(z)| > r} \frac{|g(z)|^2(\log \frac{1}{|z|})^\beta}{(1 - |\varphi(z)|^2)^2} dA(z) \leq C \int_{|\varphi(z)| > r} \frac{|g(z)|^2(1 - |z|)^{\beta}}{(1 - |\varphi(z)|^2)^2} dA(z) \]
\[ \leq C \varepsilon \int_{|\varphi(z)| > r} \frac{dA(z)}{(1 - |z|)^2} = LC \varepsilon. \]

On the other hand, we clearly have
\[ \int_{|\varphi(z)| \leq r} \frac{|g(z)|^2(1 - |z|)^{\beta}}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty. \]

Hence, \( \varphi \in D_{1, \beta, 2}^h \), and
\[ \lim_{t \to 1} \int_{|\varphi(z)| > r} \frac{|g(z)|^2(1 - |z|)^{\beta}}{(1 - |\varphi(z)|^2)^2} (\log \frac{1}{|z|})^\beta dA(z) = 0. \]

Therefore, by Theorem 2.2, \( C^\phi_\beta : B \to D^\beta_\alpha \) is compact. \( \square \)

3. The Boundedness and Compactness of
\[ C^\phi_\beta : D^\beta_\beta \to B^\alpha \]
In this section, we characterize the boundedness and compactness of the operator $C^\varphi_g$ from weighted Dirichlet type spaces $D^p_\beta$ to Bloch type spaces $B^\alpha$.

**Theorem 3.1.** Let $0 < \alpha, p < \infty$, $\beta > -1$, $g \in \mathcal{H}(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C^\varphi_g : D^p_\beta \rightarrow B^\alpha$ is bounded if and only if $L := \sup_{z \in \mathbb{D}} \frac{|g(z)|(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\frac{\beta + 2}{p}}} < \infty$.

**Proof.** For the boundedness of $C^\varphi_g$, let $f \in D^p_\beta$. Then $f' \in A^p_\beta$ and from Lemma 1.1, we have

$$|f'(z)| \leq C \frac{\|f'\|_{A^p_\beta}}{(1 - |z|^2)^{\frac{2 + \beta}{p}}} = C \frac{\|f\|_{D^p_\beta}}{(1 - |z|^2)^{\frac{2 + \beta}{p}}} \quad (z \in \mathbb{D}). \quad (6)$$

From this inequality along with $C^\varphi_g f(0) = 0$, it follows that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |(C^\varphi_g f)'(z)| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(\varphi(z))| |g(z)|$$

$$= \sup_{z \in \mathbb{D}} (1 - |\varphi(z)|^2)^{\frac{\beta + 2}{p}} \frac{(1 - |\varphi(z)|^2)^{\frac{\beta + 2}{p}} |f'(\varphi(z))|}{(1 - |\varphi(z)|^2)^{\frac{\beta + 2}{p}}}$$

$$\leq C \sup_{z \in \mathbb{D}} (1 - |\varphi(z)|^2)^\alpha |g(z)| \|f\|_{D^p_\beta}$$

$$= CL \|f\|_{D^p_\beta}.$$

Therefore, $C^\varphi_g : D^p_\beta \rightarrow B^\alpha$ is bounded.

Conversely, consider the functions $f_a(z) = \int_0^z \frac{1 - |a|^2}{(1 - \bar{a}\xi)^{\beta + 2}} d\xi$, for $a \in \mathbb{D}$. Then by Lemma 1.1, $f_a \in D^p_\beta$ and $\|f_a\|_{D^p_\beta} \simeq 1$ with constant depending only on $\beta$ and $p$. Fix $z_0 \in \mathbb{D}$ and $a = \varphi(z_0)$, the boundedness of $C^\varphi_g : D^p_\beta \rightarrow B^\alpha$ implies that $\|C^\varphi_g f_a\|_{B^\alpha} \leq C \|f_a\|_{D^p_\beta} \leq C$ for all $a$, since $C^\varphi_g f_a(0) = 0$. Therefore,
\[ C \geq \|C_\varphi^g f_a\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f_a'(\varphi(z))||g(z)| \]
\[ = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \left| \frac{1 - |a|^2}{(1 - a\varphi(z))^2} \right|^{\frac{\beta+2}{\beta}} |g(z)| \]
\[ \geq \frac{(1 - |z_0|^2)^\alpha |g(z_0)|}{(1 - |\varphi(z_0)|^2)^{\frac{\beta+2}{\beta}}}. \]

Hence, \( \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{\beta+2}{\beta}}} < \infty. \) \( \square \)

**Theorem 3.2.** Let \( 0 < \alpha, p < \infty, \beta > -1, g \in \mathcal{H}(\mathbb{D}) \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then \( C_\varphi^g : \mathcal{D}_\beta^p \rightarrow \mathcal{B}^\alpha \) is compact if and only if \( C_\varphi^g : \mathcal{D}_\beta^p \rightarrow \mathcal{B}^\alpha \) is bounded and \( \lim_{|\varphi(z)| \rightarrow 1} \frac{|g(z)(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\frac{\beta+2}{\beta}}} = 0. \)

**proof.** Let \( C_\varphi^g : \mathcal{D}_\beta^p \rightarrow \mathcal{B}^\alpha \) be compact. Note that \( f_a \), defined as in the previous theorem, converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( |a| \rightarrow 1 \). Let \( (z_n) \) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_n)| \rightarrow 1 \) as \( n \rightarrow \infty \). Using the test function \( f_n(z) = f_{\varphi(z_n)}(z) \), we get

\[ \|C_\varphi^g f_n\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |(C_\varphi^g f_n)'(z)| \]
\[ \geq (1 - |z_n|^2)^\alpha \left( \frac{1}{1 - |\varphi(z_n)|^2} \right)^{\frac{\beta+2}{\beta}} |g(z_n)|, \]

from which the result follows.

Conversely, from the boundedness of \( C_\varphi^g \) with \( f(z) = z \), we have

\[ L := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g(z)| < \infty. \]

Suppose \( (f_n) \) is a sequence in unit ball of \( \mathcal{D}_\beta^p \) such that \( f_n \rightarrow 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( n \rightarrow \infty \). By definition of limit, for every \( \varepsilon > 0 \) there is \( \delta \in (0,1) \) such that \( \sup_{\{z:|\varphi(z)|>\delta\}} \left| \frac{|g(z)(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\frac{\beta+2}{\beta}}} \right| < \varepsilon. \) Then using (6), we get
\[ \| C^g_{\varphi} f_n \|_{B^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |(C^g_{\varphi} f_n)'(z)| \]
\[ = \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^{\alpha} |f'_n(\varphi(z))||g(z)| \]
\[ + \sup_{\delta < |\varphi(z)| < 1} (1 - |z|^2)^{\alpha} |f'_n(\varphi(z))||g(z)| \]
\[ \leq L \sup_{|w| \leq \delta} |f'_n(w)| + C\| f_n \|_{D^p_\beta} \sup_{|\varphi(z)| > \delta} \frac{|g(z)|(1 - |z|^2)^{\alpha}}{(1 - |\varphi(z)|^2)^{\frac{\beta+2}{p}}} \]
\[ \leq L \sup_{|w| \leq \delta} |f'_n(w)| + C\varepsilon \| f_n \|_{D^p_\beta} \]
\[ \leq L \sup_{|w| \leq \delta} |f'_n(w)| + C\varepsilon. \]

By [3, IV, Theorem 2.1], the sequence \( (f'_n) \) also converges to zero on compact subsets of \( \mathbb{D} \) as \( n \to \infty \). In particular \( \sup_{|w| \leq \delta} |f'_n(w)| \to 0 \) as \( n \to \infty \). It follows that \( \| C^g_{\varphi} f_n \|_{B^\alpha} \to 0 \) as \( n \to \infty \). By Lemma 1.2, \( C^g_{\varphi} : D^p_\beta \to B^\alpha \) is compact. □

Finally, we give a connection between two operators \( gC_{\varphi} \) and \( C^g_{\varphi} \).

To do this, for \( \alpha > 0 \) we consider the standard weighted Banach spaces of analytic functions defined as follows:

\[ H^\infty = \{ f \in \mathcal{H}(\mathbb{D}) : \| f \|_{H^\infty} = \sup \{ (1 - |z|)^\alpha |f(z)| : |z| < 1 \} < \infty \}, \]

\[ H^0 = \{ f \in H^\infty : \lim_{|z| \to 1} (1 - |z|)^\alpha |f(z)| = 0 \}. \]

For more details about spaces of this type we refer to [1, 2, 9, 12] and the references therein. Then, we consider the differentiation operator \( D \) and the integration operator \( T \) from \( \mathcal{H}(\mathbb{D}) \) into \( \mathcal{H}(\mathbb{D}) \) defined by \( (Df)(z) = f'(z) \) and \( (Tf)(z) = \int_0^z f(\xi)d\xi \), respectively. Now, consider the following diagrams

\[
\begin{array}{ccc}
H^\infty \xrightarrow{gC_{\varphi}} A^p_\beta & & A^p_\beta \xrightarrow{gC_{\varphi}} H^\infty \\
\uparrow D & & \uparrow D \\
B^\alpha \xrightarrow{C^g_{\varphi}} D^p_\beta & & D^p_\beta \xrightarrow{C^g_{\varphi}} B^\alpha \\
\uparrow T & & \uparrow T \\
H^\infty \xrightarrow{gC_{\varphi}} A^p_\beta & & A^p_\beta \xrightarrow{gC_{\varphi}} H^\infty.
\end{array}
\]
Clearly, by the definitions of the spaces $H^\infty_\alpha$, $B^\alpha$, $A^p_\beta$ and $D^p_\beta$, the operators $D$ and $T$ between these spaces as mentioned in the above diagrams are bounded for $\alpha, p > 0$ and $\beta > -1$. Also these diagrams commute, that is,

$$T \circ gC_\varphi \circ D = C^\beta_\varphi$$

and

$$D \circ C^\beta_\varphi \circ T = gC_\varphi,$$

which imply that, weighted composition operators $gC_\varphi : A^p_\beta \rightarrow H^\infty_\alpha$ and $gC_\varphi : H^\infty_\alpha \rightarrow A^p_\beta$ are bounded (compact) if and only if the generalized composition operators $C^\beta_\varphi : D^p_\beta \rightarrow B^\alpha$ and $C^\beta_\varphi : B^\alpha \rightarrow D^p_\beta$ are bounded (compact), respectively.

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