Closability of Module $\sigma$–Derivations

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Abstract. Let $\sigma$ be a linear mapping from a dense subalgebra $A$ of a Banach algebra $B$ into $B$. In this note, we study the closability of a module $\sigma$– derivation $\delta$ from $A$ into a $B$– bimodule $M$. Applying the notions of torsion-free modules and essential ideals, we present several results concerning the closability of such derivations. Also we investigate the closability of module $\sigma$– derivations of the $C^*$– algebra $B$ into a Hilbert $B$– bimodule $M$.

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1. Introduction

Throughout the paper, $A$ is a dense subalgebra of a Banach algebra $B$ and $M$ is a Banach $B$– bimodule. We recall that a linear mapping $\delta: A \rightarrow M$ is a (module) derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. A derivation $\delta$ is said to be inner if there exists an element $u \in M$ such that $\delta(a) := ua - au$, for all $a \in A$. Recently, a number of

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analysts have studied various generalized notions of derivations in the context of Banach algebras. As an example, suppose that \( \sigma : A \to B \) is a homomorphism. If for every \( u \in M \), we take \( \delta_\sigma^u : A \to M \) by \( \delta_\sigma^u(a) := u\sigma(a) - \sigma(a)u \), then it is easily seen that \( \delta_\sigma^u(ab) = \delta_\sigma^u(a)\sigma(b) + \sigma(a)\delta_\sigma^u(b) \) for all \( a, b \in A \). Therefore considering the relation \( \delta(ab) = \delta(a)b + a\delta(b) \) as an special case of \( \delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b) \) for all \( a, b \in A \), where \( \sigma : A \to B \) is a linear mapping, leads the theory of derivations to be extensively developed.

The above consideration motivated the authors in [8,9] to generalize the notion of derivation as follows:

Let \( \sigma : A \to B \) be a linear mapping. By a (module) \( \sigma - \) derivation we mean a linear mapping \( \delta : A \to M \) such that \( \delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b) \) for all \( a, b \in A \). In order to construct a \( \sigma - \) derivation, suppose that \( u \) is an element of \( M \) satisfying

\[
 u(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))u.
\]

Then the mapping \( \delta_u^\sigma \) defined by \( \delta_u^\sigma(a) := u\sigma(a) - \sigma(a)u \) is a module \( \sigma - \) derivation which is called inner. Note that if \( \sigma \) is an endomorphism, then \( u \) can be any arbitrary unitary element of \( M \). It is easy to see that if \( \sigma \) is bounded, then the module \( \sigma - \) derivation \( \delta_u^\sigma \) is bounded. The reader is referred to [5,8,9,10] for more details on \( \sigma - \) derivations.

A linear mapping \( \delta : A \to M \) is called closable if it has a closed linear extension. For a linear mapping \( \delta : A \to M \), we let \( S(\delta) \) denote the set

\[
 \{ x \in M : \text{there is a sequence } \{a_n\} \text{ in } A \text{ with } a_n \to 0 \text{ and } \delta(a_n) \to x \}
\]

and call it the separating space of \( \delta \). \( \delta \) is closable iff \( S(\delta) = \{0\} \). It is obvious that if \( \delta \) is continuous, then it is closable but the converse does not hold in general. We refer the reader to [4,13,14] for more information on the concept of closability. In this note as a main result we show that if \( \sigma \) is a continuous surjective linear mapping and \( \delta \) is a module \( \sigma - \) derivation, then the separating space \( S(\delta) \) is bimodule and applying this result we conclude the closability of a \( \sigma - \) derivation \( \delta \) under some restrictions on the codimensions of the sets \( \{a \pm \delta(a) : a \in A\} \) which are called the deficiency indices.

Let \( \delta_0 : A \to M \) be a linear mapping. Following [14], a module \( \sigma - \) derivation \( \delta \) is called relative bounded with respect to \( \delta_0 \) (or briefly \( \delta_0 - \)
bounded) if there exist $\alpha, \beta > 0$ such that $\| \delta(a) \| \leq \alpha \| a \| + \beta \| \delta_0(a) \|$, for all $a \in A$. Among other facts we show that for a linear operator $\delta_0 : A \to M$ and a $\delta_0$-bounded module $\sigma$-derivation $\delta$ if there exists a core $D$ for $\delta_0$ such that the restriction of $\delta$ on $D$ is closable, then $\delta$ is closable.

For an element $a$ in a unital Banach algebra $A$, let $sp(a)$ be the set of all complex number $\lambda$ such that $\lambda - a$ is not invertible in $A$ and call it the spectrum of $a$. The spectral radius of $a$ is defined by $\nu(a) := \sup \{ | \lambda | : \lambda \in sp(a) \}$. An element $a$ is called quasi-nilpotent if $\nu(a) = 0$. The set of all quasi-nilpotents is denoted by $Q(A)$. An algebra $A$ is called semi-simple if $rad(A) = \{0\}$, where $rad(A)$ is defined to be the intersection of the maximal ideals in $A$, ([See 3]).

Let $B$ be a $C^*$-algebra and $M$ be a complex linear space which is a left $B$-module and $\lambda(bx) = (\lambda b)x = b(\lambda x)$, where $\lambda \in \mathbb{C}$, $b \in B$ and $x \in M$. The space $M$ is called a left pre-Hilbert $B$-module, if there exists a $B$-valued inner product $<, > : M \times M \to B$ such that for every $x, y, z \in M$, $\lambda \in \mathbb{C}$ and $b \in B$, satisfies the following conditions:

\begin{enumerate}
  \item $(i) < x, x > \geq 0$
  \item $(ii) < x, x > = 0$ if and only if $x = 0$
  \item $(iii) < x + \lambda y, z > = < x, z > + \lambda < y, z >$
  \item $(iv) < x, y > = < y, x >^*$
  \item $(v) < ax, y > = a < x, y >$
\end{enumerate}

Similarly, we can define a right pre-Hilbert $B$-module. The left (right) pre-Hilbert $B$-module $M$ is called Hilbert $B$-module if it is a Banach space with respect to the norm $\| x \| := \| < x, x > \|^2$. The Hilbert module $M$ is called full if the closed linear span $< M, M >$ of all elements of the form $< x, y >$ $(x, y \in M)$ is equal to $B$. Let $M$ be a right pre-Hilbert $B$-module with the inner product $<, >_1$ and a left pre-Hilbert $B$-module with the inner product $<, >_2$. Then $M$ is a pre-Hilbert $B$-bimodule if for every $x, y, z \in M$ and for each $a, b \in B$, the following conditions hold:

\begin{enumerate}
  \item $(i) < x, y >_2 z = x < y, z >_1$
  \item $(ii) < bx, bx >_1 \leq \| b \|^2 < x, x >_1$ and $< xa, xa >_2 \leq \| a \|^2 < x, x >_2$
\end{enumerate}

In [7] it is shown that if $M$ is a pre-Hilbert $B$-bimodule, then
∥x∥:=∥<x,x>1∥1/2=∥<x,x>2∥1/2 defines a norm on M. We also investigate the closability of module σ− derivations from a dense subalgebra A of a C∗− algebra B into Hilbert B− bimodule M.

2. The Results

Theorem 2.1. Let \( \delta : A \to M \) be a bounded below module σ− derivation such that \( S(\delta) = R(\delta) \). Then \( \delta = 0 \). In particular, \( \delta \) is closable.

Proof. Let \( x \in S(\delta) \). Then there exists a sequence \( \{a_n\} \) in A such that \( a_n \to 0 \) and \( \delta(a_n) \to x \). Since \( x \in S(\delta) = R(\delta) \), so \( \delta(a) = x \) for some \( a \in A \). Also \( \delta \) is bounded below hence there exists \( C > 0 \) such that \( C \|a\| \leq \|\delta(a)\| \) for all \( a \in A \). This implies that \( \delta \) is an injection and \( \delta^{-1} \) is bounded. Therefore \( a_n \to \delta^{-1}(x) = a \). But \( a_n \to 0 \) thus \( a = 0 \) and \( x = \delta(a) = 0 \).

Theorem 2.2. Let \( M \) be a simple B− bimodule in the sense that it has no non-trivial two-sided submodule, σ : A → B be a surjective continuous linear mapping and let \( \delta : A \to M \) be a module σ− derivation. Then either \( \delta \) is closable or the range \( R(\delta) \) of \( \delta \) is dense in \( M \).

Proof. It is obvious that \( S(\delta) \) is a closed subspace of \( M \). We show that \( S(\delta) \) is a two-sided submodule of \( M \). Let \( b \in B \) and \( x \in S(\delta) \). Thus there is a sequence \( \{a_n\} \) in A such that \( a_n \to 0 \) and \( \delta(a_n) \to x \). Since \( \sigma \) is a surjection, so there exists \( c \in A \) such that \( \sigma(c) = b \). Hence \( ca_n \to 0 \) and by continuity of \( \sigma \) we have \( \delta(ca_n) = \sigma(c)\delta(a_n) + \delta(c)\sigma(a_n) \to bx \). Thus \( bx \in S(\delta) \). A similar argument shows that \( xb \in S(\delta) \). By the hypothesis \( S(\delta) = \{0\} \) or \( S(\delta) = M \). Therefore \( \delta \) is closable or the range of \( \delta \) is dense in \( M \).

Since every simple Banach algebra \( B \) is itself a simple A− bimodule, we have the two following results.

Corollary 2.3. Let \( A \) be a dense subalgebra of a simple Banach algebra \( B \), σ : A → B be a surjective continuous linear mapping and let \( \delta : A \to B \) be a σ− derivation. Then either \( \delta \) is closable or both of the sets \( \{a \pm \delta(a) : a \in A\} \) are dense in \( B \).
Proof. Following as stated in the proof of Theorem 2.2, one can observe that $S(\delta)$ is a two-sided ideal in $B$. If $S(\delta) = \{0\}$, then $\delta$ is closable. In the case that $S(\delta) = B$, then $R(\delta)$ is dense in $B$. Hence both of the sets \{\(a \pm \delta(a) : a \in A\)\} are dense in $B$. □

Corollary 2.4. Let $A$ be a dense subalgebra of a simple Banach algebra $B$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a $\sigma-$ derivation such that the set \{\(a + \delta(a) : a \in A\)\} is closed. Then either $\delta$ is closable or the map from $A$ into $B$ which takes $a \mapsto a + \delta(a)$ is onto.

Proof. Follows from the Corollary 2.3.

The proof of the following result is exactly similar to the method has been used in [13]. □

Theorem 2.5. Let $A$ be a dense subalgebra of a simple unital Banach algebra $(B, \| \cdot \|)$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a $\sigma-$ derivation. Suppose that $(A, | \cdot |)$ is a Banach algebra for some norm $| \cdot |$, defined on the domain $A$ of $\delta$ such that $\delta : (A, | \cdot |) \to (B, \| \cdot \|)$ is continuous. If the deficiency indices of $\delta$ are finite and not equal, then $\delta : A \to B$ is closable.

Remark 2.6. Let $A$ be a dense subalgebra of a simple $C^*$ algebra $(B, \| \cdot \|)$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta, \delta_0 : A \to B$ be $\sigma-$ derivations such that $\delta_0$ is closed and $\delta : (A, | \cdot |) \to (B, \| \cdot \|)$ is continuous, where the norm $| \cdot |$ is defined by $| a | = \| a \| + \| \delta_0(a) \|$. Since $\| a \| \leq | a |$, it follows that $I(a)$ is continuous and therefore $I \pm \delta$ are continuous maps from $(A, | \cdot |)$ into $(B, \| \cdot \|)$. If one of the deficiency indices of $\delta$ is finite and non-zero, then one of the two sets \{\(a \pm \delta(a) : a \in A\)\} is closed and not equal to $B$. Using Corollary 2.4, we conclude that $\delta$ is closable.

Before we state the next theorem, we need the following useful lemma which can be found in [3].

Lemma 2.7. Let $A$ be a unital Banach algebra and $I$ be an ideal in $A$ with $I \subseteq Q(A)$. Then $I \subseteq \text{rad}(A)$. (See [3], Proposition 2.2.3, p 16).

Using the concept of semi-simplicity and the above lemma, we have the
Theorem 2.8. Let $A$ be a dense subalgebra of a unital semi-simple Banach algebra $B$, $\sigma : A \rightarrow B$ be a surjective continuous linear mapping and let $\delta : A \rightarrow B$ be a $\sigma-$ derivation. If $S(\delta)$ is contained in the set of quasi nilpotent elements of $B$, then $\delta$ is closable.

Proof. The method has been used in the proof of Theorem 2.2 shows that $S(\delta)$ is a two-sided ideal in $B$ and by our assumption $S(\delta) \subseteq Q(B)$. Thus $S(\delta)$ is contained in the radical of $B$. The semi-simplicity of $B$ implies that $S(\delta) = \{0\}$. Hence $\delta$ is closable. □

Following the argument as stated in [14], we have the next two results.

Theorem 2.9. Let $\delta, \delta_0 : A \rightarrow M$ be module $\sigma-$ derivations such that $\delta_0$ is closable and $\delta-$ bounded. If $\delta$ is $\delta_0-$ bounded, then is closable.

Proof. Let $x \in S(\delta)$. Thus there is a sequence $\{a_n\}$ in $A$ such that $a_n \rightarrow 0$ and $\delta(a_n) \rightarrow x$. Since $\delta_0$ is $\delta-$ bounded, so there exists a real number $\alpha > 0$ such that

$$\| \delta_0(a_n) - \delta_0(a_m) \| = \| \delta_0(a_n - a_m) \| \leq \alpha (\| a_n - a_m \| + \| \delta(a_n) - \delta(a_m) \|)$$

and

$$\alpha (\| a_n - a_m \| + \| \delta(a_n) - \delta(a_m) \|) \rightarrow 0 \quad (as \quad m, n \rightarrow \infty).$$

Thus $\{\delta_0(a_n)\}$ is a Cauchy sequence in the Banach $B-$ module $M$ and hence is convergent. Because of the closability of $\delta_0$, we have $\delta_0(a_n) \rightarrow 0$. On the other hand since $\delta$ is $\delta_0-$ bounded, so there exists a real number $\beta > 0$ such that

$$\| \delta(a_n) \| \leq \beta (\| a_n \| + \| \delta_0(a_n) \|) \rightarrow 0 \quad (as \quad n \rightarrow \infty).$$

Therefore $x = 0$ and hence $\delta$ is closable. □

Remark 2.10. Let $\delta, \delta_0 : A \rightarrow M$ be module $\sigma-$ derivations such that $\delta_0$ is closable and $\delta-$ bounded. Suppose that $\delta - \delta_0$ is $\delta-$ bounded. Hence
there exists two positive numbers $\alpha, \beta$ such that

$$\| (\delta - \delta_0)(a) \| \leq \alpha \| a \| + \beta \| \delta(a) \| .$$

Therefore

$$\| \delta(a) \| - \| \delta_0(a) \| \leq \alpha \| a \| + \beta \| \delta(a) \| .$$

An easy computation shows that if $1 - \beta > 0$, then $\delta$ is $\delta_0$-bounded and by the above theorem, $\delta$ is closable.

Before we state the next theorem, we recall the following well-known definition.

**Definition 2.11.** A subset $D$ of domain $D(\delta_0)$ is called a core for $\delta_0$, if $\delta_0$ is the closure of its restriction on $D$.

**Theorem 2.12.** Let $\delta_0 : A \to M$ be a linear operator and $\delta : A \to M$ be a $\delta_0$-bounded module $\sigma$-derivation. If there exists a core $D$ for $\delta_0$ such that the restriction $\delta \mid_D : D \to M$ is closable, then $\delta$ is closable.

**Proof.** First note that since $\delta$ is $\delta_0$-bounded, so there exists a positive number $\beta$ such that

$$\| \delta(a) \| \leq \beta(\| a \| + \| \delta_0(a) \|).$$

Let $x \in S(\delta)$. Thus there is a sequence $\{a_n\}$ in $A$ such that $a_n \to 0$ and $\delta(a_n) \to x$. Let $n \in \mathbb{N}$. Since the subset $D$ of $A$ is a core for $\delta_0$, so there exists a sequence $\{c^n_k\}$ in $D$ such that $c^n_k \to a_n$ and $\delta_0(c^n_k) \to \delta_0(a_n)$. Hence for every fixed $n$ there exist a positive integer $N$ such that $\| c^n_N - a_n \| < \frac{1}{2^n}$ and $\| \delta_0(c^n_N) - \delta_0(a_n) \| < \frac{1}{2^n}$. Because of the $\delta_0$-boundedness of $\delta$ we have

$$\| \delta(c^n_N) - \delta(a_n) \| \leq \beta(\| c^n_N - a_n \| + \| \delta_0(c^n_N - a_n) \|) < \frac{\beta}{n}.$$

Thus

$$\| \delta(c^n_N) - x \| \leq \| \delta(c^n_N) - \delta(a_n) \| + \| \delta(a_n) - x \| \to 0.$$
That is \( \delta(c^N_M) \rightarrow x \). But \( \delta |_D: D \rightarrow M \) is closable, therefore \( x = 0 \).  \( \square \)

Before we state the next theorem, we recall the following well-known definition.

**Definition 2.13.** Let \( x \) be in an \( A \)-bimodule \( M \). The annihilator \( x^\perp \) of \( x \) is defined by \( x^\perp := \{ a \in A : ax = 0 \} \). Then \( A \)-bimodule \( M \) is called torsion-free if the torsion submodule \( M_t := \{ x \in M : x^\perp \neq \{0\} \} \) be zero. (i.e. for each \( x \in M - \{0\} \), \( x^\perp = \{0\} \).)

**Theorem 2.14.** Let \( I \) be a non-zero ideal in a dense subalgebra \( A \) of a Banach algebra \( B \), \( \sigma : A \rightarrow B \) be a surjective continuous linear mapping satisfying \( \sigma(I) \neq \{0\} \). Suppose that \( \delta : A \rightarrow M \) is a module \( \sigma \)-derivation such that the restriction \( \delta |_I: I \rightarrow M \) is closable. If \( S(\delta) \) is a torsion-free module, then \( \delta \) is closable.

**Proof.** First note that the surjectivity of \( \sigma \) implies that \( S(\delta) \) is a submodule of \( M \). Let \( x \in S(\delta) \). Then there exists a sequence \( \{a_n\} \) in \( A \) such that \( a_n \rightarrow 0 \) and \( \delta(a_n) \rightarrow x \). It is enough to show that \( x^\perp \neq \{0\} \). For this, let \( a \) be a non-zero element in \( I \) such that \( \sigma(a) \neq 0 \). Then \( aa_n \rightarrow 0 \) and by continuity of \( \sigma \) we have \( \delta(aa_n) \rightarrow \sigma(a)x \). But \( aa_n \in I \) and by the assumption the restriction of \( \delta \) on \( I \) is closable, so \( \sigma(a)x = 0 \). This shows that \( x^\perp \) contains a non-zero element \( \sigma(a) \). Now since \( S(\delta) \) is torsion-free, hence \( x = 0 \).  \( \square \)

**Theorem 2.15.** Let \( A \) be a dense subalgebra of a Banach algebra \( B \), \( M \) be a torsion-free \( B \)-bimodule and let \( \delta : A \rightarrow M \) be a non-zero continuous module \( \sigma \)-derivation. Then \( \sigma \) is closable.

**Proof.** Let \( b \in S(\sigma) \). Then there exists a sequence \( \{a_n\} \) in \( A \) such that \( a_n \rightarrow 0 \) and \( \sigma(a_n) \rightarrow b \). Let \( x \) be a non-zero element in \( R(\delta) \). So there exists a non-zero element \( a \in A \) such that \( \delta(a) = x \). Hence \( a_n a \rightarrow 0 \) and by continuity of \( \delta \) we have \( \delta(a_n a) \rightarrow b\delta(a) = bx \). Using the continuity of \( \delta \) once more, we have \( bx = 0 \). This shows that \( b \in x^\perp \) and since \( M \) is torsion-free, hence \( b = 0 \).  \( \square \)

**Definition 2.16.** An ideal \( I \) in an algebra \( B \) is called essential if its annihilator \( I^\perp := \{ b \in B : bI = \{0\} \} \) is zero.
Replacing the module $M$ by the Banach algebras $B$ and using the concept of the "essential ideal", we have the following:

**Theorem 2.17.** Let $A$ be a dense subalgebra of a Banach algebra $B$, $I$ be an essential ideal of $B$ which is contained in $A$, $\sigma : A \to B$ be a continuous linear mapping such that $\{0\} \neq \sigma(I) \subseteq I$ and $\delta : A \to B$ be a $\sigma-$ derivation such that the restriction $\delta |_I : I \to B$ is closable, then $\delta$ is closable.

**Proof.** Let $b \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in $A$ such that $a_n \to 0$ and $\delta(a_n) \to b$. Let $a$ be a non-zero element in $I$ such that $\sigma(a) \neq 0$. Hence $aa_n \to 0$ and by continuity of $\sigma$ we have $\delta(aa_n) \to \sigma(a)b$. Because of the closability of the restriction $\delta |_I$ we have $\sigma(a)b = 0$. But $I$ is an essential ideal of $B$ so $b = 0$. \(\square\)

**Theorem 2.18.** Let $M$ be a $B-$ bimodule with an approximate identity $\{e_i\}$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to M$ be a module $\sigma-$ derivation. If for every ideal $(a)$ generated by $a \in A$, the restriction $\delta |_{(a)} : (a) \to M$ is closable, then $\delta$ is closable.

**Proof.** Let $x \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in $A$ such that $a_n \to 0$ and $\delta(a_n) \to x$. Let $a \in A$. Since $\sigma$ is surjective, there exists an element $c \in A$ such that $\sigma(c) = a$. Then $ca_n \to 0$ and by continuity of $\sigma$ we have $\delta(ca_n) \to \sigma(c)x = ax$. But $ca_n \in (c)$ and by assumption the restriction of $\delta$ on $(c)$ is closable, so $ax = 0$, for every $a \in A$. The density of $A$ in $B$ implies that $bx = 0$, for every $b \in B$. Since $e_i \in B$, then $e_ix = 0$. But $e_ix \to x$, hence $x = 0$. \(\square\)

The following results concentrate on the closability of module $\sigma-$ derivations in Hilbert $C^*$- modules:

Let $A$ be a dense subalgebra of a $C^*$- algebra $B$, $M$ a Hilbert $B-$ module and let $\{e_i\}$ be an approximate identity for $B$. We have:

$$< x - e_i x, x - e_i x> = < x, x > - e_i < x, x > + e_i < x, x > e_i - < x, x > e_i$$

Hence $< x - e_i x, x - e_i x> \to 0$. Therefore $e_ix \to x$. So by the Theorem 2.18 we have the next corollary.
Corollary 2.19. Let $A$ be a dense subalgebra of a $C^*$-- algebra $B$, $M$ a Hilbert $B$-- module, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to M$ be a module $\sigma$-- derivation. If for every ideal $(a)$ generated by $a \in A$, the restriction $\delta \mid_{(a)} : (a) \to M$ is closable, then $\delta$ is closable.

Corollary 2.20. Let $A$ be a dense subalgebra of a $C^*$-- algebra $B$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a module $\sigma$-- derivation. If for every ideal $(a)$ generated by $a \in A$, the restriction $\delta \mid_{(a)} : (a) \to B$ is closable, then $\delta$ is closable.

Before we state the next theorem, we need the following useful lemma which can be found in [11].

Lemma 2.21. Let $M$ be a full Hilbert $B$-- module and $b \in B$. If $bx = 0$, for every $x \in M$ then $b = 0$.

Theorem 2.22. Let $A$ be a dense subalgebra of a $C^*$-- algebra $B$, $M$ be a full Hilbert $B$-- bimodule and let $\delta : A \to M$ be a surjective continuous module $\sigma$-- derivation. Then $\sigma$ is closable.

Proof. Let $b \in S(\sigma)$. Then there exists a sequence $\{a_n\}$ in $A$ such that $a_n \to 0$ and $\sigma(a_n) \to b$. Let $x \in M$. Since $\delta$ is surjective, so there exists an element $a \in A$ such that $\delta(a) = x$. Hence $a_n a \to 0$ and by continuity of $\delta$ we have $\delta(a_n a) \to b\delta(a) = bx$. Using the continuity of $\delta$ once more, we have $bx = 0$, for all $x \in M$. But $M$ is full and by the previous lemma, we have $b = 0$. \[\square\]

Before we state the next theorem, we need the following useful lemma which can be found in [1].

Lemma 2.23. Let $I$ be an ideal in a $C^*$-- algebra $B$. The following conditions are mutually equivalent:

(i) $I$ is an essential ideal in $B$;

(ii) $||c|| = \sup_{b \in I, ||b|| \leq 1} ||cb||$, $\forall c \in B$;

(iii) $||c|| = \sup_{b \in I, ||b|| \leq 1} ||bc||$, $\forall c \in B$;

(iv) $||c|| = \sup_{b \in I, ||b|| \leq 1} ||bcb^*||$, $\forall c \in B$. 

Theorem 2.24. Let $A$ be a dense subalgebra of a $C^*$—algebra $B$, $I$ be an essential ideal of $B$ which is contained in $A$, $M$ be a Hilbert $B$—bimodule and let $\sigma : A \to B$ be a continuous linear mapping such that $\sigma(I) = I$. If $\delta : A \to M$ is a module $\sigma$—derivation such that the restriction $\delta \mid_I : I \to M$ is closable, then $\delta$ is closable.

Proof. Let $x \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in $A$ such that $a_n \to 0$ and $\delta(a_n) \to x$. It is enough to show that $\|x\| = 0$. For this, let $b$ be a non-zero element in $I$ satisfying $\|b\| \leq 1$. Since $\sigma(I) = I$, so there exists a non-zero element $a \in I$ such that $\sigma(a) = b$. Then $aa_n \to 0$ and by continuity of $\sigma$ we have $\delta(aa_n) \to \sigma(a)x = bx$. But $aa_n \in I$ and by assumption the restriction of $\delta$ on $I$ is closable, so $bx = 0$. The fact that $I$ is an essential ideal of $B$ and the above lemma implies that

$$\|x\|^2 = \|\langle x, x \rangle\| = \sup_{b \in I, \|b\| \leq 1} \|b \langle x, x \rangle b^*\| = \sup_{b \in I, \|b\| \leq 1} \|\langle bx, bx \rangle\| = 0.$$  

The following is an immediate consequence of Theorem 2.24.

Corollary 2.25. Let $A$ be a dense subalgebra of a $C^*$—algebra $B$, $I$ be a non-zero ideal of $B$ which is contained in $A$, $M$ be a Hilbert $B$—bimodule such that $\|x\| := \sup_{b \in I, \|b\| \leq 1} \|\langle bx, bx \rangle\|$ holds for all of $x \in M$ and let $\sigma : A \to B$ be a continuous linear mapping such that $\sigma(I) = I$. If $\delta : A \to M$ is a module $\sigma$—derivation such that the restriction $\delta \mid_I : I \to M$ is closable, then $\delta$ is closable.

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