

Remarks On $\aleph_0$ -Injectivity

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Abstract. This article is a continuation of E. Momtahan's work in which he studied the literature via countable injectivity. Along this line, $\aleph_0$ -Ikeda-Nakayama rings have been studied and among other things, new characterizations of basically disconnected and extremally disconnected spaces have been given. Furthermore we will observe that a boolean ring is $\aleph_0$ -self-injective if and only if it is an $\aleph_0$ -IN ring.

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1. Introduction

In this article, by R we mean an associative ring with identity and M is a unitary R -module. An R -module M is said to be $\aleph_0$ -injective (f-injective) if for every (module) homomorphism $f \in \text{Hom}_R(I, M)$, there exists $\bar{f} \in \text{Hom}_R(R, M)$, such that $\bar{f}|_I = f$, where I is any countably generated (finitely generated) right ideal of R . A ring R is said to be right $\aleph_0$ -self-injective (f-injective) if it is $\aleph_0$ -injective (f-injective) as a right R -module. In this note, by $C(X)$ we mean the ring of all real valued continuous functions over a Hausdorff completely regular space (or equivalently a Tychonoff space). The reader is referred to [8] for undefined notations and definitions. A module is called extending ($\aleph_0$ -extending) if every submodule (countably generated submodule) is essential in a direct summand.

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This paper is a continuation of [15] and [16], in which the author studied $\aleph_0$ -injectivity of modules and rings. In [15], many well-known observations on injective modules have been seen to be true if we replace injectivity by $\aleph_0$ -injectivity. Among them, for example, one can quote, this one: a ring R is semisimple artinian if and only every R -module (or every countably generated R -module) is $\aleph_0$ -injective. Using this result, one can prove the following proposition which is well-known if we replace injectivity by $\aleph_0$ -injectivity. The equivalence of the first two parts of the following result is due to B. Osofsky (see [14, page 114, excersise 11]).

Proposition 1.1. *The following assertions are equivalent:*

1. R is semisimple Artinian.
2. the intersection of any two injective R -modules is injective.
3. the intersection of any two $\aleph_0$ -injective R -modules is $\aleph_0$ -injective.

Proof. The proof of $(1) \Leftrightarrow (3)$ works also for $(1) \Leftrightarrow (2)$. $(1) \Rightarrow (3)$ is obvious. Suppose that the intersection of any two $\aleph_0$ -injective R -modules is $\aleph_0$ -injective. Now suppose that R were not semisimple Artinian. In [15], it has been observed that a ring is semisimple Artinian if and only if every module is $\aleph_0$ -injective. Hence by our assumption there is an R -module M which is not $\aleph_0$ -injective. Set $E = E(M) \oplus E(E(M)/M)$. Then E is injective. Let x belong to $E(M) \setminus M$. Set $y = (x, x+M) \in E$. Then $(M, 0)$ is essential in $(E(M), 0)$ and $E((M, 0) + yR) \subseteq E$ and the intersection of those two injectives is M which is not $\aleph_0$ -injective. A contradiction. $\square$

Remark 1.2. *It is well-known that a ring is von Neumann regular if and only if every module over R is f -injective. Now the same method used in the above result shows that a ring R is von Neumann regular if and only if the intersection of any two f -injective module is f -injective. It is also well-known that a ring R is (left and right) artinian serial with $\text{Jac}(R)^2 = 0$ if and only if every module over R is extending (see [6, 13.5]). Again, the same line of proof shows that (ii) a ring R is (left and right) artinian serial if and only if the intersection of any two extending modules is extending.*

However, if every finitely generated (or even cyclic) R -module is $\aleph_0$ -injective, we do not need to access the semisimplicity of R (contrary to Osofsky's famous result which asserts that a ring R is semisimple artinian if and only if every cyclic R -module is injective). In fact, a ring is right $\aleph_0$ -self-injective regular if and only if every cyclic R -module is $\aleph_0$ -injective. In [16], it has been found out that the well-known result of Y. Utumi that *every right (or left) self-injective ring modulo its Jacobson radical is von Neumann regular*, is no longer true if we replace injectivity by $\aleph_0$ -injectivity. A commutative $\aleph_0$ -self-injective ring with many additional properties has been constructed that is not von Neumann regular modulo its Jacobson radical.

The next proposition will help us in the sequel. It is well-known for injectivity (see [22]).

Proposition 1.3. *Let M be an R -module and $I \subseteq \text{Ann}(M)$. If M is an $\aleph_0$ -injective R -module, then M is an $\aleph_0$ -injective R/I -module. The converse is true provided that R is a right fully idempotent ring.*

Proof. Let M be an $\aleph_0$ -injective R -module, and $f : A/I \rightarrow M$ be an R/I -homomorphism, where A/I is a countably generated right ideal. Thus we may put $A = B + I$, where $B = \sum_{k=1}^{\infty} a_k R$ and then let $f(a_k + I) = m_k$, where $m_k \in M$. Defining $g : B \rightarrow M$ by $g(a_k) = m_k = f(a_k + I)$, by our hypothesis g can be extended to R . Suppose now M is $\aleph_0$ -injective as an R/I -module, and let $f : A \rightarrow M$ be an R -homomorphism, where A is a countably generated right ideal. First we claim that $f(A \cap I) = 0$: for every $x \in A \cap I$ we have $x = \sum_{k=1}^{k=n} a_k b_k$ where a_i and b_i belong to $A \cap I$ and therefore

$$f(x) = f\left(\sum_{k=1}^{k=n} a_k b_k\right) = \sum_{k=1}^{k=n} f(a_k) b_k$$

but $f(a_k) b_k \in MI = 0$ so $f(A \cap I) = 0$. In as much as $(A + I)/I \cong A/(A \cap I)$ and $f(A \cap I) = 0$, we infer that $\bar{f} : (A + I)/I \rightarrow M$ is an R/I -module homomorphism so $\bar{f} : R/I \rightarrow M$, i.e., $\bar{f}(a + I) = (a + I)(r + I) = ar + I$ for some $r \in R$, i.e. $f(a) = ar$. $\square$

2. $\aleph_0$ -Ikeda-Nakayama Rings

In [11, Theorem 1], it has been proved that a ring R is right f-injective if and only if (i) $l\text{Ann}(I_1 \cap I_2) = l\text{Ann}(I_1) + l\text{Ann}(I_2)$, where I_1 and I_2 are finitely generated right ideals of R and (ii) $r\text{Ann}(l\text{Ann}(I)) = I$ where I is a principal ideal right ideal of R . Rings satisfy in (i), for each pair of right ideals are called Ikeda-Nakayama (IN-)rings. Such rings have been extensively studied (see for example [4]). IN-rings have been shown to be a special class of quasi-continuous rings (see [4], where it has been observed for the first time). Furthermore, the theory of IN-modules has been introduced and studied in [21]. It is easy to see that every right self-injective ring is a right IN-ring. However, the converse is not true. This is because, the ring of integers, $\mathbb{Z}$, is an IN-ring that is not self-injective, since $\mathbb{Z}$ is not a divisible abelian group. Here we consider $\aleph_0$ -IN rings and study their behaviour in Boolean rings and rings of real valued continuous functions over a Tychonoff space.

Definition 2.1. *A ring is called a right $\aleph_0$ -IN ring if $l\text{Ann}(I_1 \cap I_2) = l\text{Ann}(I_1) + l\text{Ann}(I_2)$, where I_1 and I_2 are countably generated right ideals of R .*

We begin with a lemma which is useful in the sequel:

Lemma 2.2. *If R is a commutative $\aleph_0$ -selfinjective ring, then R is an $\aleph_0$ -IN ring.*

Proof. Obviously $l\text{Ann}(I_1) + l\text{Ann}(I_2) \subseteq l\text{Ann}(I_1 \cap I_2)$. On the other hand, suppose that $a \in l\text{Ann}(I_1 \cap I_2)$, we can define a homomorphism $\alpha : I_1 + I_2 \longrightarrow R$ as:

$$\alpha(b) = \begin{cases} b & b \in I_1, \\ (1 + a)b & b \in I_2. \end{cases}$$

Since these two expressions coincide on $I_1 \cap I_2$, by $\aleph_0$ -injectivity of R , $\exists c \in R$ such that $\alpha(b) = cb$ for $b \in I_1$. Thus, we have $cb = b$, i.e., $(c - 1)b = 0$. Consequently we write $a = (c - 1) + (1 + a - c)(c - 1) \in l\text{Ann}(I_1)$ and $(1 + a - c) \in l\text{Ann}(I_2)$, this proves the lemma.

For the sequel we need the following result which is a slight modification

of Theorem 8 in [21]. $\square$

Lemma 2.3. *Let M_R be a right R -module and $S = \text{End}(M)$. Then the following are equivalent:*

1. M is $\aleph_0$ - π -injective ($\aleph_0$ -quasi continuous)
2. For any two countably generated submodules A and B of M_R , $S = l\text{Ann}_S(A) + l\text{Ann}_S(B)$.

Using these results we have the following result in $C(X)$. In the following a space X is called *extremally disconnected* if open sets have open closure. By a *basically disconnected* space we mean a space in which co-zero sets (i.e. the complement of zero sets) have open closure, where by a zero set of $f \in C(X)$ we mean $f^{-1}\{0\} = \{x \in X : f(x) = 0\}$.

Theorem 2.4. *Let X be a completely regular space. Then the following are equivalent:*

1. for any two countably generated ideals A and B of $C(X)$, with $A \cap B = (0)$;

$$\text{Ann}(A) + \text{Ann}(B) = C(X)$$
2. X is basically disconnected.

Proof. (1) $\Rightarrow$ (2) By Lemma 2.3, we observe that $C(X)$ is $\aleph_0$ -(quasi)-continuous. Hence it is an $\aleph_0$ -extending ring. By [3, Theorem 3.3], X is basically disconnected.

(2) $\Rightarrow$ (1) Since X is basically disconnected, again by [3], Theorem 3.3, we observe that $C(X)$ is $\aleph_0$ -extending. Let $\langle e_1 \rangle$ and $\langle e_2 \rangle$ be two ideals which are summands of a commutative ring R . It is well-known that $\langle e_1 \rangle + \langle e_2 \rangle = \langle e_1 + e_2 - e_1 e_2 \rangle$ and $\langle e_1 \rangle \cap \langle e_2 \rangle = \langle e_1 e_2 \rangle$. Hence $C(X)$ satisfies summand sum property (SSP) and summand intersection property. This implies that $C(X)$ is an $\aleph_0$ -extending ring if and only if $C(X)$ is an $\aleph_0$ -quasi continuous ring. $\square$

Theorem 2.5. *Let X be a completely regular space. Then the following are equivalent:*

1. for any two ideals A and B of $C(X)$ with $A \cap B = (0)$,

$$\text{Ann}(A) + \text{Ann}(B) = C(X)$$

2. X is an extremally disconnected space.

Proof. (1) $\Rightarrow$ (2) By [21], Theorem 8, $C(X)$ is a quasi-continuous ring and hence extending. By [3], Theorem 3.5, we deduce that X is extremally disconnected.

(2) $\Rightarrow$ (1) By [3], Theorem 3.5. when X is extremally disconnected, then $C(X)$ is an extending ring. But as we stated in the proof of the above theorem, $C(X)$ has always summand sum property. Therefore it is quasi-continuous when it is extending. Now by [21, Theorem 8] the implication follows. This completes the proof. $\square$

Remark 2.6. *It is natural to speculate on the space X if $C(X)$ satisfies the following stronger conditions: (i) for any two ideals A and B of $C(X)$, $\text{Ann}(A) + \text{Ann}(B) = C(X)$, and (ii) for any two countably generated ideals A and B of $C(X)$, $\text{Ann}(A) + \text{Ann}(B) = C(X)$. We conjecture that in the first case X is an extremally P -space (and hence $C(X)$ is self-injective regular in this case) and in the second case, X would be a P -space. In [7], it has been shown that $C(X)$ is $\aleph_0$ -self-injective if and only if $C(X)$ is regular. Hence if our conjecture (i.e., conjecture (ii)) is true, we would have: $C(X)$ is $\aleph_0$ -IN if and only if $C(X)$ is $\aleph_0$ -self-injective.*

The following result is due to O. A. S. Karamzadeh and A. A. Koochakpour ([13]). Let A and B be two subsets of the ring R . We say $A \cup B$ is an *orthogonal set* if for every x, y in $A \cup B$ with $x \neq y$, we have $xy = 0$. We say that an element x separates A from B if $xa^2 = a$ for all $a \in A$ and $xB = 0$. If there exists such an x , then we say that A has a left separation from B .

Lemma 2.7. *For a strongly regular ring R the following are equivalent:*

1. R is $\aleph_0$ -self-injective

2. If $S \cup T$ is a countable orthogonal set in R with $S \cap T = \emptyset$, then S has a left separation from T .

Andrew B. Carson [5], has shown that if R is an $\aleph_0$ -complete Boolean ring (Boolean joins of countably many elements always exist), then R is $\aleph_0$ -self-injective. He also showed that there are $\aleph_0$ -self-injective Boolean rings which are not $\aleph_0$ -complete. In spite of this fact, we show that in a Boolean ring the following fact holds:

Proposition 2.8. *Let R be a Boolean ring, then the following are equivalent:*

1. R is an $\aleph_0$ -IN-ring.
2. for any two countably generated ideals A and B of R with $A \cap B = (0)$, $\text{Ann}(A) + \text{Ann}(B) = R$.
3. R is $\aleph_0$ -self-injective.

Proof. (1) $\Rightarrow$ (2) It is always true.

(2) $\Rightarrow$ (3) We want to show that R is an $\aleph_0$ -self-injective ring. According to Lemma 2.7, it is enough to show that each two disjoint orthogonal countable subsets of R can be separated by an element of R . Let $S \cup T$ be an orthogonal set with $S \cap T = \emptyset$. Since R is an $\aleph_0$ -IN ring, we have

$$\text{Ann}(\langle S \rangle \cap \langle T \rangle) = \text{Ann}(\langle S \rangle) + \text{Ann}(\langle T \rangle)$$

It is then evident that $R = \text{Ann}(\langle S \rangle \cap \langle T \rangle) = \text{Ann}(\langle S \rangle) + \text{Ann}(\langle T \rangle)$, i.e., $1 = x + y$, where $xS = 0$ and $yT = 0$. Now we see that $1 - x = y$ separates S from T , because $yT = 0$ and $yS = (1 - x)S = S$ i.e., $ya = (1 - x)a = a - xa = a$. But R is a Boolean ring, hence $ya^2 = a$.

(3) $\Rightarrow$ (1) Follows from Lemma 2.2. $\square$

Along this line, we observe that a Boolean ring is an IN-ring if, and only if, it is self-injective. We need the following lemma which has been proved in [13]:

Lemma 2.9. *For a strongly regular ring R the following are equivalent:*

1. R is self-injective
2. If $S \cup T$ is an orthogonal set in R with $S \cap T = \emptyset$, then S has a left separation from T .

Proposition 2.10. *Let R be a Boolean ring, then the following are equivalent:*

1. R is an IN-ring,
2. for any two ideals A and B of R with $A \cap B = (0)$, $\text{Ann}(A) + \text{Ann}(B) = R$.
3. R is self-injective.

proof. Use Lemma 2.9 and follow exactly the proof of Proposition 2.8. $\square$

3. $\sum$ - $\aleph_0$ -Injectivity

In [9], C. Faith has shown that a $\sum$ -injective R -module M with endomorphism ring S is characterized by the ascending chain condition on the lattice of S -submodules which are annihilators of subsets of R ([9, Proposition 3.3]). If $E(R)$ denotes the injective hull of R_R , and if $M = E(R)$, this condition implies the ascending chain condition on annihilator right ideals (=right annulets) of R , and in case $M = E(R) = R$, this condition is equivalent to a.c.c. on right annulets ([9, Corollary 3.4 and Theorem 3.5]). Now we observe that some parts of these results by C. Faith in [9] can be applied to answer a similar question concerning $\sum$ - $\aleph_0$ -injectivity. Let M_R be a left R -module and $S = \text{End}(M)$. Following Faith's nomenclature, for any subset X of M ,

$$X^\perp = \text{Ann}_R(X) = \{r \in R \mid Xr = 0\}$$

is a right ideal of R . The set of such right ideals is denoted by $\mathcal{A}$. By $\mathcal{A}_{\aleph_0}$, we mean all countably generated members of $\mathcal{A}$. And by $\mathcal{B}_{\aleph_0}$, we mean

the set of all annihilators of members of $\mathcal{A}_{\aleph_0}$ in M . Since $I \longrightarrow I^\perp$ is an injective and order-inverting map between $\mathcal{A}_{\aleph_0}$ and $\mathcal{B}_{\aleph_0}$, one satisfies the ascending chain condition if, and only if, the other one satisfies the descending chain condition.

Proposition 3.1. *If M_R is an R -module, then $\mathcal{A}_{\aleph_0}$ satisfies the ascending chain condition if and only if for each countably generated right ideal I of R there exists a finitely generated subideal I_1 such that $I^\perp = I_1^\perp$.*

Proof. Assume a.c.c. for $\mathcal{A}_{\aleph_0}$ or equivalently, the d.c.c. for $\mathcal{B}_{\aleph_0}$. Let I be a countably generated right ideal of R , and let I_1 be a finitely generated subideal of I , such that $I_1^\perp$ be a minimal element of the set

$$\{K^\perp : K \text{ is a finitely generated subideal of } I\}$$

in M . If $x \in I$, then $I_1 + xR$ is also a finitely generated subideal of I , and hence satisfies $(I_1 + xR)^\perp \subseteq I_1^\perp$. By the choice of I_1 , necessarily $(I_1 + xR)^\perp = I_1^\perp$, so $I_1^\perp x = 0$. Since this is true for all $x \in I$, then $I_1^\perp I = 0$, whence, $I_1^\perp \subseteq I^\perp$. Moreover, $I_1 \subseteq I$ implies $I^\perp \subseteq I_1^\perp$, consequently $I_1^\perp = I^\perp$ follows.

Conversely, let $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ be a chain of countably generated right ideals of R lying in $\mathcal{A}_{\aleph_0}$, let $X_i = I_i^\perp$, $i = 1, 2, \dots$, be the corresponding elements of $\mathcal{B}_{\aleph_0}$, and suppose also $I = \bigcup_{n=1}^\infty I_n$. Now, let J be the finitely generated subideal of I such that $I^\perp = J^\perp$. Since J is finitely generated, there is an integer q such that $J \subseteq I_k$, $k \geq q$. Thus, $I_k^\perp \subseteq J^\perp$, $k \geq q$. Moreover,

$$J^\perp = I^\perp = \bigcap_{n=1}^\infty I_n^\perp,$$

Consequently, $I_k^\perp = J^\perp$, for $k \geq q$. Then, $I_k = (I_k^\perp)^\perp = I_q$, $k \geq q$, and the result follows. $\square$

Corollary 3.2. *A ring R satisfies the a.c.c. on $\mathcal{B}_{\aleph_0}$ if and only if each countably generated right ideal I contains a finitely generated right ideal I_1 , such that $I^\perp = I_1^\perp$.*

Proposition 3.3. *The following conditions on an $\aleph_0$ -injective module M are equivalent:*

1. $M^{(\mathbb{N})}$ is $\aleph_0$ -injective.
2. R satisfies the a.c.c. on the ideals in $\mathcal{A}_{\aleph_0}$.
3. M is $\sum$ - $\aleph_0$ -injective.

Proof. (1) $\Rightarrow$ (2) Let $I_1 \subset I_2 \subset \cdots \subset I_m \subset \cdots$ be a strictly ascending chain of right ideals in $\mathcal{A}_{\aleph_0}$, let $I = \bigcup_{n=1}^{\infty} I_n$, and let x_n , be an element of $I_n^{\perp}$ (taken in M) not in $I_{n+1}^{\perp}$, $n = 1, 2, \dots$. If $r \in I$, then $x_n r = 0$ for all $n \geq q$. Therefore the element $r' = (x_1 r, \dots, x_n r, \dots)$ lies in $M^{(\mathbb{N})}$, even though $x = (x_1, \dots, x_n, \dots)$ lies in $M^{\mathbb{N}}$. Let f denote a map defined by $f(r) = r'$ for all $r \in I$. Assuming that $M^{(\mathbb{N})}$ is injective, there exists, by Baer's criterion, an element $y = (y_1, \dots, y_m, 0, \dots) \in M^{(\mathbb{N})}$ such that

$$f(r) = yr = (y_1 r, \dots, y_m r, 0, \dots) = (x_1 r, \dots, x_m r, \dots)$$

for all $r \in I$. But this implies that $x_t r = 0$ for all $t > m$, for all $r \in I$, that is, $x_t \in I^{\perp} \subseteq I_{t+1}^{\perp}$, this contradicts the choice of x_t .

(2) $\Rightarrow$ (3). Let I be a right ideal of R , and $I_1 = r_1 R + \cdots + r_n R$ be the finitely generated subideal given by the above proposition such that $I^{\perp} = I_1^{\perp}$. Let $f : I \rightarrow M^{(\Lambda)}$ be any map. Since M^{Λ} is injective (it is direct product of injective modules), there exists an element $p \in M^{\Lambda}$ such that $f(r) = pr$ for all $r \in I$. Since $f(r_i) = pr_i \in M^{(\Lambda)}$, $i = 1, \dots, n$, there exists an element $p' \in M^{(\Lambda)}$ such that $p_a r_i = p'_a r_i$ for all $a \in \Lambda$, $i = 1, \dots, n$, where g_a is the a coordinate of any $g \in M^{\Lambda}$. Since $r_1, \dots, r_n$ generate I_1 , this implies that $pr = p'r$ for all $r \in I_1$, hence $(p_a - p'_a) \in I_1^{\perp}$ for all $a \in \Lambda$. Since $I_1^{\perp} = I^{\perp}$, it follows that for all $x \in I$ and for all $a \in \Lambda$, that is $px = p'x$, for all $x \in I$. Thus, $f(x) = p'x$ for all $x \in I$, with $p' \in M^{(\Lambda)}$, so $M^{(\Lambda)}$ is $\aleph_0$ -injective. $\square$

Along this line and as an application of Proposition 1.3, we consider the following theorem which is well-known when we replace $\aleph_0$ -injectivity by injectivity. We need the following lemma.

Lemma 3.4. *Suppose R is a regular ring and M is an R -module. Then M is $\sum$ - $\aleph_0$ -injective if and only if R/A is an artinian ring where $A = \text{Ann}(M)$.*

Proof. A standard proof given in [19], works here with only a slight modification. $\square$

Theorem 3.5. *Let R be a regular ring which is an algebra over a field F . Let M be an R -module with $\dim_F M \leq \aleph_0$, and let $I = \text{Ann}(M)$. Then the following are equivalent:*

1. R/I is Artinian.
2. M is $\sum$ - $\aleph_0$ -injective.
3. M is $\aleph_0$ -injective.

Proof. Using the proof of the above theorem (see [19]) and apply Proposition 1.3. $\square$

4. On $\aleph_0$ -Quasi Injectivity

It is well-known that $\aleph_0$ -injectivity is a Morita property for regular rings ([10, Chapter 14]). In fact, by a theorem of D. Handelman we have: Let R be a right $\aleph_0$ -self-injective regular ring. If M is a finitely generated projective right R -module, then $T = \text{End}(M)$ is a right $\aleph_0$ -self-injective regular ring. We say that a module is $\aleph_0$ -quasi injective if every R -homomorphism $f : B \rightarrow M$ extends to M , when B is a countably generated submodule of M . By the proof of Handelman's theorem we infer that:

Corollary 4.1. *Let R be an $\aleph_0$ -selfinjective regular ring, and P be a finitely generated projective module, then P is $\aleph_0$ -quasi injective.*

As we saw over $\aleph_0$ -self-injective regular rings, every finitely generated free module is $\aleph_0$ -quasi injective. This is a good motivation to study $\aleph_0$ -quasi injectivity. J. Ahsan [2] introduced the concept of qc-ring. Here we will present results on $\aleph_0$ -qc ring

Definition 4.2. *A ring R is called right $\aleph_0$ -qc ring if every R -homomorphic image of R as a right R -module is $\aleph_0$ -quasi injective.*

The next result is an extension of a theorem of Nicholson and Yousif [19]. Although it is stated for quasi injective module, with only a slight modification, it is true for countably generated $\aleph_0$ -quasi injective modules. Recall that a module M is Dedekind-finite if $M \oplus N \cong M$ then $N = 0$. Remind that a module M is called co-hopfian if every injective f in $\text{End}(M)$ is an isomorphism.

Proposition 4.3. *Let M be a quasi injective module. Then M is Dedekind-finite if, and only if, M is co-hopfian.*

Proof. It is well-known that M is Dedekind-finite if, and only if, $\text{End}(M)$ is a Dedekind-finite ring. Now let $f : M \longrightarrow M$ be an R -homomorphism. If f is a monomorphism, then there exists $g : f(M) \longrightarrow M$ such that $g = f^{-1}$, but $f(M)$ is a submodule of M , so there exists $g' : M \longrightarrow M$ such that $g'f = 1$. But $\text{End}(M)$ is Dedekind-finite so $fg' = 1$, whence, f is an epimorphism. Conversely, let $fg = 1_M$. This implies that g is monic ($g(m) = 0 \Rightarrow fg(m) = 1(m) = m = 0$). Since M is co-hopfian, g is an isomorphism, hence $g = f^{-1}$. $\square$

Lemma 4.4. *Let R be a ring, and I an ideal of R . If M is an $\aleph_0$ -quasi injective R/I -module, then M is also $\aleph_0$ -quasi injective as an R -module. Also, if M is an $\aleph_0$ -quasi injective R -module and $MI = 0$, then M is also $\aleph_0$ -quasi injective as an R/I -module.*

Proof. The relation $(*)$: $m(r + I) = mr$ ($m \in M$ and $r \in R$) is used in each case to define M as a module over R or R/I , where M is given as a module over R/I or R . Consider that if $N = \langle x_1, x_2, \dots \rangle$ is countably generated as an R/I -module, then for all $x \in N$ there exist $i_1, i_2, \dots, i_k$ such that $x = x_{i_1}(r_{i_1} + I) + \dots + x_{i_n}(r_{i_n} + I)$.

By $(*)$, $x = x_{i_1}r_{i_1} + \dots + x_{i_n}r_{i_n}$, this means that N is also countably generated as an R -module. It is easy to see that the concepts "submodule" and "homomorphism" coincide over each ring. Hence any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & M \\ & & \downarrow & & \\ & & M & & \end{array}$$

over any ring is also a diagram over the other ring. Thus M is $\aleph_0$ -quasi-injective over R if, and only if, M is $\aleph_0$ -quasi-injective over R/I . The result follows. $\square$

Lemma 4.5. *Let R be a ring and K an arbitrary two sided ideal of R . If R is a right $\aleph_0$ -qc ring then R/K is a right $\aleph_0$ -qc ring.*

Proof. Suppose R is a right $\aleph_0$ -qc ring and K is a two sided ideal of R . We show that R/K is a right $\aleph_0$ -qc ring. Let I/K be any right ideal of R/K with $I \subseteq R$ and consider the right R/K -module $R/K/I/K \cong R/I$ as an R -module with $K \subseteq I$. Then K annihilates the R -module R/I , and therefore, R/I may be regarded as an R/K -module. Furthermore, R is a right $\aleph_0$ -qc ring by hypothesis. Hence, R/I is R - $\aleph_0$ -quasi-injective. Hence, by the above lemma, R/I , as an R/K -module, is R/K - $\aleph_0$ -quasi-injective. We have shown that any cyclic R/K -module is R/K - $\aleph_0$ -quasi-injective. Therefore R/K is a right $\aleph_0$ -qc ring. $\square$

Theorem 4.6. *Let R be a commutative ring. Then R is an $\aleph_0$ -qc ring if, and only if, every factor ring of R is an $\aleph_0$ -self-injective ring.*

Proof. If I is an ideal of an $\aleph_0$ -qc ring R , then R/I is an $\aleph_0$ -qc ring by lemma 4.5. Therefore R/I is an $\aleph_0$ -self-injective ring (it is evident that since R_R is generated by the identity, we may infer that any homomorphism from countably generated ideal of R into R_R can be extended to an endomorphism). Hence R_R is $\aleph_0$ -injective, and R is a cyclic R -module. Then, $M \cong R/I$ for some ideal I of R . By assumption, R/I is R/I - $\aleph_0$ -quasi-injective. Hence R/I is R - $\aleph_0$ -quasi-injective, and therefore R is an $\aleph_0$ -qc ring. Now suppose that I is an ideal of R . Then R/I is $\aleph_0$ -self-injective ring if and only if R/I is $\aleph_0$ -quasi-injective as an R -module. $\square$

5. Examples

In this section we provide some new examples of $\aleph_0$ -self-injective (regular) rings which are perhaps of some interest for their own right.

Example 5.1. Let F be a field and G a group. If $\text{char}(F) = 0$, then

the following are equivalent:

1. $F[G]$ is $\aleph_0$ -self-injective.
2. G is finite.
3. $F[G]$ is self-injective.

Proof. (1) $\Rightarrow$ (2) Suppose $F[G]$ is $\aleph_0$ -self-injective, then it is p -injective (here p -injective means principally injective). By a result of Farkas ([19]), G is locally finite. Now by a result of Villamayor-Connell [19], page 69, Theorem 1.5, $F[G]$ is a regular ring. The rest of proof is the same as Renault's Theorem ([19], Theorem 2.8). The remaining implications are well-known and can be found in [19]. $\square$

Example 5.2. Let X be a Tychonoff space. In [7], it has been shown that $C(X)$ is $\aleph_0$ -self-injective if, and only if, $C(X)/C_F(X)$ is $\aleph_0$ -self-injective if, and only if, X is a P-space and therefore, if, and only if, $C(X)$ is regular.

Example 5.3. Let $\mathcal{A}$ be a σ -Algebra. In [1], it has been observed that rings of all real valued $\mathcal{A}$ -measurable functions are $\aleph_0$ -self-injective.

Example 5.4. Let X be Tychonoff space. By $D(X)$, we mean the lattice of continuous functions f on X with values in the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$, for which $f^{-1}(\mathbb{R})$ is a dense subset of X . In general, under pointwise addition and multiplication, $D(X)$ is not a ring. However, when X is a quasi-F space, then $D(X)$ is a ring. A Tychonoff space X is called a quasi F-space if every dense co-zero set S in X is C^* -embedded. Since $D(X) \cong D(\beta X)$, we may without loss of generality suppose that X is a compact space. Then by a result due to Hager (see [1]), $D(X) \cong \frac{M}{N}$, for some certain N . This then implies that $D(X)$ is also an $\aleph_0$ -self-injective regular ring, for X a quasi-F space (see [1]).

Example 5.5. Let X be the one point compactification of the discrete space D of cardinal ω_1 . Let $S(X)$, the ring of continuous real-valued functions f on X with $f(X)$ finite. It can be seen easily that R is regular and $\aleph_0$ -self-injective.

The following example is a slight modification of a theorem by F. L. Sandomierski (see [20]).

Example 5.6. Let M be a right R -module and a direct sum of countably many non-zero submodules $\{M_n \mid n \in \mathbb{N}\}$ and $S = \text{End}(M)$. Then ${}_S M$ is not an $\aleph_0$ -injective S -module.

Proof. Let $e_i : M_R \rightarrow M_i$ be the i -th projection of the module M_R onto the submodule M_i , then $\{e_i\}_{i \in \mathbb{N}}$ is a countable set of orthogonal idempotents of S . Let ${}_S A$ be the ideal of S generated by $\{e_i\}_{i \in \mathbb{N}}$. Since $M_i \neq 0$ for each $i \in \mathbb{N}$, choose $0 \neq x_i \in M_i$. Clearly there is an S -homomorphism $f : {}_S A \rightarrow {}_S M$ such that $e_i f = x_i = e_i x_i$. If f were extendable to a homomorphism from S to ${}_S M$, then it would be given by some element of M . However, for any element $x \in M$, $e_i x = 0$ for all but finitely many $i \in \mathbb{N}$, so f is not extendable to ${}_S S$, so ${}_S M$ is not $\aleph_0$ -injective. $\square$

Example 5.7. Let $R = \mathbb{Z}(+)\mathbb{Q}$ be a ring with addition $(m, p) + (n, q) = (m + n, p + q)$ and multiplication $(m, p)(n, q) = (mn, mq + np)$. The ring R is called the idealization of $\mathbb{Z}$ with respect to the $\mathbb{Z}$ -module $\mathbb{Z}$. Now put $S = R^\omega$, that is $S = \prod_{i \in \mathbb{N}} R_i$ where each R_i is a copy of R , clearly $S \cong \mathbb{Z}^\omega(+)(\mathbb{Q})^\omega$. Let W be the ideal $(\mathbb{Z}(+)\mathbb{Q})^{(\omega)}$ of S , the restriction of the isomorphism $S \cong \mathbb{Z}^\omega(+)(\mathbb{Q})^\omega$ to W is an isomorphism $W \cong \mathbb{Z}^{(\omega)}(+)(\mathbb{Q})^{(\omega)}$. Put $S' := S/W \cong \mathbb{Z}^\omega(+)(\mathbb{Q})^\omega / \mathbb{Z}^{(\omega)}(+)(\mathbb{Q})^{(\omega)} \cong \mathbb{Z}^\omega / \mathbb{Z}^{(\omega)}(+)(\mathbb{Q})^\omega / (\mathbb{Q})^{(\omega)}$. In [16], it has been observed that S' is an $\aleph_0$ -self-injective ring whereas $S'/J(S')$ is not a regular ring.

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